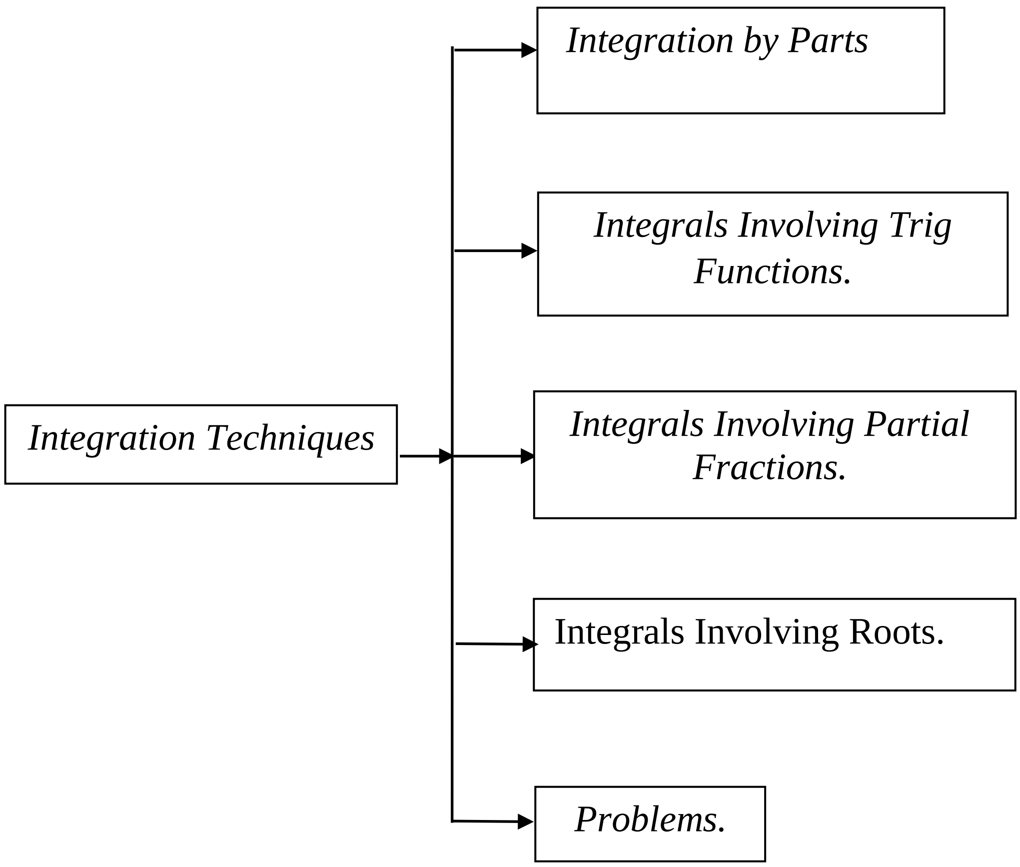
Lecture – One

Integration Techniques



Integration Techniques:-

1.1 Integration by Parts

So let's derive the integration by parts formula. We'll start with the product rule.

$$(f\,g)' = f'\,g + f\,g'$$

Now, integrate both sides of this.

$$\int (f\,g)'\,dx = \int f'\,g + f\,g'\,dx$$

The left side is easy enough to integrate and we'll split up the right side of the integral.

$$fg = \int f'\,g\,dx + \int f\,g'\,dx$$

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

$$\int f\,g'\,dx = fg - \int f'\,g\,dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$u = f(x) \qquad\qquad v = g(x)$$

$$du = f'(x)\,dx \qquad\qquad dv = g'(x)\,dx$$

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

$$\int u\,dv = uv - \int v\,du$$

To use this formula we will need to identify u and dv, compute du and v and then use the formula. Note as well that computing v is very easy. All we need to do is integrate dv.

$$v = \int dv$$

Example 1 Evaluate the following integral

$$\int x\mathbf{e}^{6x}\,dx$$

Solution

$$u = x \qquad\qquad dv = \mathbf{e}^{6x}\,dx$$

$$du = dx \qquad\qquad v = \int \mathbf{e}^{6x}\,dx = \frac{1}{6}\mathbf{e}^{6x}$$

The integral is then,

$$\int x\mathbf{e}^{6x}\,dx = \frac{x}{6}\mathbf{e}^{6x} - \int \frac{1}{6}\mathbf{e}^{6x}\,dx$$
$$= \frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x} + c$$

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

Integration by Parts, Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 2 Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

Solution

This is the same integral that we looked at in the first example so we'll use the same u and dv to get,

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx = \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2}\mathbf{e}^{6x}\,dx$$
$$= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{36}\mathbf{e}^{6x}\Big|_{-1}^{2}$$
$$= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}$$

Example 3 Evaluate the following integral.

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

Solution

Instead of splitting the integral up let's instead use the following choices for u and dv.

$$u = 3t+5 \qquad dv = \cos\left(\frac{t}{4}\right)dt$$

$$du = 3\,dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

The integral is then,

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt = 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt$$
$$= 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + c$$

Example 4 Evaluate the following integral.

$$\int w^2 \sin(10w)\,dw$$

Solution

For this example we'll use the following choices for u and dv.

$$u = w^2 \qquad dv = \sin(10w)\,dw$$

$$du = 2w\,dw \qquad v = -\frac{1}{10}\cos(10w)$$

The integral is then,

$$\int w^2 \sin(10w)\,dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\int w\cos(10w)\,dw$$

In this example, unlike the previous examples, the new integral will also require integration by parts. For this second integral we will use the following choices.

$$u = w \qquad dv = \cos(10w)\,dw$$

$$du = dw \qquad v = \frac{1}{10}\sin(10w)$$

So, the integral becomes,

$$\int w^2 \sin(10w)\,dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) - \frac{1}{10}\int \sin(10w)\,dw\right)$$

$$= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) + \frac{1}{100}\cos(10w)\right) + c$$

$$= -\frac{w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + c$$

Example 5 Evaluate the following integral

$$\int x\sqrt{x+1}\,dx$$

(a) Using Integration by Parts. [Solution]
(b) Using a standard Calculus I substitution. [Solution]

Solution

(a) Evaluate using Integration by Parts.

In this case we'll use the following choices for u and dv.

$$u = x \qquad dv = \sqrt{x+1}\,dx$$

$$du = dx \qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

The integral is then,

$$\int x\sqrt{x+1}\,dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c$$

(b) Evaluate Using a standard Calculus I substitution.

Now let's do the integral with a substitution. We can use the following substitution.

$$u = x+1 \qquad x = u-1 \qquad du = dx$$

Notice that we'll actually use the substitution twice, once for the quantity under the square root and once for the x in front of the square root. The integral is then,

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Example 6 Evaluate the following integral.

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta$$

Solution

$$u = \cos\theta \qquad dv = \mathbf{e}^{\theta}\, d\theta$$
$$du = -\sin\theta\, d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is then,

$$\int \mathbf{e}^{\theta} \cos\theta\, d\theta = \mathbf{e}^{\theta}\cos\theta + \int \mathbf{e}^{\theta}\sin\theta\, d\theta$$

So, it looks like we'll do integration by parts again. Here are our choices this time.

$$u = \sin\theta \qquad dv = \mathbf{e}^{\theta}\, d\theta$$
$$du = \cos\theta\, d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is now,

$$\int \mathbf{e}^{\theta} \cos\theta\, d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta - \int \mathbf{e}^{\theta}\cos\theta\, d\theta$$
$$2\int \mathbf{e}^{\theta} \cos\theta\, d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta$$

All we need to do now is divide by 2 and we're done. The integral is,

$$\int \mathbf{e}^{\theta} \cos\theta\, d\theta = \frac{1}{2}\left(\mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta\right) + c$$

1.2 ***Integrals Involving Trig Functions***.

Let's start off with an integral that we should already be able to do.

$$\int \cos x \sin^5 x\, dx = \int u^5\, du \qquad \text{using the substitution } u = \sin x$$
$$= \frac{1}{6}\sin^6 x + c$$

Example 1 Evaluate the following integral.

$$\int \sin^5 x \, dx$$

Solution

This integral no longer has the cosine in it that would allow us to use the substitution that we used above. Therefore, that substitution won't work and we are going to have to find another way of doing this integral.

Let's first notice that we could write the integral as follows,

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int \left(\sin^2 x\right)^2 \sin x \, dx$$

Now recall the trig identity,

$$\cos^2 x + \sin^2 x = 1 \qquad \Rightarrow \qquad \sin^2 x = 1 - \cos^2 x$$

With this identity the integral can be written as,

$$\int \sin^5 x \, dx = \int \left(1 - \cos^2 x\right)^2 \sin x \, dx$$

and we can now use the substitution $u = \cos x$. Doing this gives us,

$$\begin{aligned}\int \sin^5 x \, dx &= -\int \left(1 - u^2\right)^2 du \\ &= -\int 1 - 2u^2 + u^4 \, du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c\end{aligned}$$

The exponent on the remaining sines will then be even and we can easily convert the remaining sines to cosines using the identity,

$$\cos^2 x + \sin^2 x = 1 \qquad (1)$$

Example 2 Evaluate the following integral.

$$\int \sin^6 x \cos^3 x \, dx$$

Solution

So, in this case we've got both sines and cosines in the problem and in this case the exponent on the sine is even while the exponent on the cosine is odd. So, we can use a similar technique in this integral. This time we'll strip out a cosine and convert the rest to sines.

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$$

$$= \int \sin^6 x \left(1 - \sin^2 x\right) \cos x \, dx \qquad u = \sin x$$

$$= \int u^6 \left(1 - u^2\right) du$$

$$= \int u^6 - u^8 \, du$$

$$= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + c$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine.

$$\int \sin^n x \cos^m x \, dx \qquad (2)$$

The integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the following formulas to rewrite the integrand.

$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$$

$$\sin x \cos x = \frac{1}{2}\sin(2x)$$

Example 3 Evaluate the following integral.

$$\int \sin^2 x \cos^2 x \, dx$$

Solution

Solution 1

In this solution we will use the two half angle formulas above and just substitute them into the integral.

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}\left(1 - \cos(2x)\right)\left(\frac{1}{2}\right)\left(1 + \cos(2x)\right) dx$$

$$= \frac{1}{4}\int 1 - \cos^2(2x) \, dx$$

In fact to eliminate the remaining problem term all that we need to do is reuse the first half angle formula given above.

$$\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int 1 - \frac{1}{2}\left(1 + \cos(4x)\right)dx$$
$$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos(4x)\,dx$$
$$= \frac{1}{4}\left(\frac{1}{2}x - \frac{1}{8}\sin(4x)\right) + c$$
$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$$

Solution 2

In this solution we will use the half angle formula to help simplify the integral as follows.

$$\int \sin^2 x \cos^2 x\,dx = \int (\sin x \cos x)^2\,dx$$
$$= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx$$
$$= \frac{1}{4}\int \sin^2(2x)\,dx$$

Now, we use the double angle formula for sine to reduce to an integral that we can do.

$$\int \sin^2 x \cos^2 x\,dx = \frac{1}{8}\int 1 - \cos(4x)\,dx$$
$$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c$$

Sometimes in the process of reducing integrals in which both exponents are even we will run across products of sine and cosine in which the arguments are different. These will require one of the following formulas to reduce the products to integrals that we can do.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

Example 4 Evaluate the following integral.

$$\int \cos(15x)\cos(4x)\,dx$$

Solution

This integral requires the last formula listed above.

$$\int \cos(15x)\cos(4x)\,dx = \frac{1}{2}\int \cos(11x)+\cos(19x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{11}\sin(11x)+\frac{1}{19}\sin(19x)\right)+c$$

It's now time to look at integrals that involve products of secants and tangents. This time, let's do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int \sec^n x \tan^m x\,dx \qquad (3)$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using a formula similar to (1). In fact, the formula can be derived from (1) so let's do that.

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x \qquad (4)$$

Now, we're going to want to deal with (3) similarly to how we dealt with (2). We'll want to eventually use one of the following substitutions.

$$u = \tan x \qquad du = \sec^2 x\,dx$$

$$u = \sec x \qquad du = \sec x \tan x\,dx$$

Example 5 Evaluate the following integral.

$$\int \sec^9 x \tan^5 x \, dx$$

Solution

First note that since the exponent on the secant isn't even we can't use the substitution $u = \tan x$. However, the exponent on the tangent is odd and we've got a secant in the integral and so we will be able to use the substitution $u = \sec x$. This means stripping out a single tangent (along with a secant) and converting the remaining tangents to secants using (4).

Here's the work for this integral.

$$\begin{aligned}\int \sec^9 x \tan^5 x \, dx &= \int \sec^8 x \tan^4 x \tan x \sec x \, dx \\ &= \int \sec^8 x \left(\sec^2 x - 1\right)^2 \tan x \sec x \, dx \qquad u = \sec x \\ &= \int u^8 \left(u^2 - 1\right)^2 du \\ &= \int u^{12} - 2u^{10} + u^8 \, du \\ &= \frac{1}{13}\sec^{13} x - \frac{2}{11}\sec^{11} x + \frac{1}{9}\sec^9 x + c\end{aligned}$$

Example 6 Evaluate the following integral.

$$\int \sec^4 x \tan^6 x \, dx$$

Solution

So, in this example the exponent on the tangent is even so the substitution $u = \sec x$ won't work. The exponent on the secant is even and so we can use the substitution $u = \tan x$ for this integral. That means that we need to strip out two secants and convert the rest to tangents. Here is the work for this integral.

$$\begin{aligned}\int \sec^4 x \tan^6 x \, dx &= \int \sec^2 x \tan^6 x \sec^2 x \, dx \\ &= \int \left(\tan^2 x + 1\right) \tan^6 x \sec^2 x \, dx \qquad u = \tan x \\ &= \int \left(u^2 + 1\right) u^6 \, du \\ &= \int u^8 + u^6 \, du \\ &= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + c\end{aligned}$$

Example 7 Evaluate the following integral.

$$\int \tan x \, dx$$

Solution
To do this integral all we need to do is recall the definition of tangent in terms of sine and cosine and then this integral is nothing more than a Calculus I substitution.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \qquad u = \cos x$$
$$= -\int \frac{1}{u} du$$
$$= -\ln|\cos x| + c \qquad r \ln x = \ln x^r$$
$$= \ln|\cos x|^{-1} + c$$
$$\ln|\sec x| + c$$

Example 8 Evaluate the following integral.

$$\int \tan^3 x \, dx$$

Solution
The trick to this one is do the following manipulation of the integrand.

$$\int \tan^3 x \, dx = \int \tan x \tan^2 x \, dx$$
$$= \int \tan x \left(\sec^2 x - 1\right) dx$$
$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

We can now use the substitution $u = \tan x$ on the first integral and the results from the previous example on the second integral.

The integral is then,

$$\int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + c$$

Note that all odd powers of tangent (with the exception of the first power) can be integrated using the same method we used in the previous example. For instance,

$$\int \tan^5 x \, dx = \int \tan^3 x \left(\sec^2 x - 1\right) dx = \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

Example 9 Evaluate the following integral.

$$\int \sec x \, dx$$

Solution

This one isn't too bad once you see what you've got to do. By itself the integral can't be done. However, if we manipulate the integrand as follows we can do it.

$$\int \sec x \, dx = \int \frac{\sec x \left(\sec x + \tan x\right)}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} dx$$

In this form we can do the integral using the substitution $u = \sec x + \tan x$. Doing this gives,

$$\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + c$$

Example 10 Evaluate the following integral.

$$\int \sec^3 x \, dx$$

Solution

This one is different from any of the other integrals that we've done in this section. The first step to doing this integral is to perform integration by parts using the following choices for u and dv.

$$u = \sec x \qquad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \qquad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

To do this integral we'll first write the tangents in the integral in terms of secants. Again, this is not necessarily an obvious choice but it's what we need to do in this case.

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

the first integral is exactly the integral we're being asked to evaluate with a minus sign in front. So, add it to both sides to get,

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

Finally divide by two and we're done.

$$\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right|\right) + c$$