## *1.3 **Integrals Involving Partial Fractions.***

let's start this section out with an integral that we can already do so we can contrast it with the integrals that we'll be doing in this section.

$$\int \frac{2x-1}{x^2-x-6}\,dx = \int \frac{1}{u}\,du \qquad \text{using} \quad u = x^2-x-6 \quad \text{and} \quad du = (2x-1)\,dx$$

$$= \ln\left|x^2-x-6\right| + c$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator) doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

| Factor in denominator | Term in partial fraction decomposition |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^k$ | $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}, \quad k=1,2,3,\ldots$ |
| $ax^2+bx+c$ | $\frac{Ax+B}{ax^2+bx+c}$ |
| $(ax^2+bx+c)^k$ | $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad k=1,2,3,\ldots$ |

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.
Let's start the examples by doing the integral above.

***Example 1*** Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

***Solution***

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose $A$ and $B$ so that the numerators of these two are equal for every $x$. To do this we'll need to set the numerators equal.

$$3x+11=A(x+2)+B(x-3)$$

What we're going to do here is to notice that the numerators must be equal for *any* $x$ that we would choose to use. In particular the numerators must be equal for $x=-2$ and $x=3$. So, let's plug these in and see what we get.

$$x=-2 \qquad 5=A(0)+B(-5) \qquad \Rightarrow \qquad B=-1$$
$$x=3 \qquad 20=A(5)+B(0) \qquad \Rightarrow \qquad A=4$$

At this point there really isn't a whole lot to do other than the integral.

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3} - \frac{1}{x+2}\,dx \\ &= \int \frac{4}{x-3}\,dx - \int \frac{1}{x+2}\,dx \\ &= 4\ln|x-3| - \ln|x+2| + c\end{aligned}$$

There is also another integral that often shows up in these kinds of problems so we may as well give the formula for it here since we are already on the subject.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

***Example 2*** Evaluate the following integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$$

***Solution***

We won't be putting as much detail into this solution as we did in the previous example. The first thing is to factor the denominator and get the form of the partial fraction decomposition.

$$\frac{x^2+4}{x(x+2)(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2}$$

The next step is to set numerators equal. If you need to actually add the right side together to get

$$x^2+4 = A(x+2)(3x-2) + Bx(3x-2) + Cx(x+2)$$

As with the previous example it looks like we can just pick a few values of $x$ and find the constants so let's do that.

$$\begin{aligned} &x=0 && 4 = A(2)(-2) && \Rightarrow && A=-1 \\ &x=-2 && 8 = B(-2)(-8) && \Rightarrow && B=\frac{1}{2} \\ &x=\frac{2}{3} && \frac{40}{9} = C\left(\frac{2}{3}\right)\left(\frac{8}{3}\right) && \Rightarrow && C=\frac{40}{16}=\frac{5}{2}\end{aligned}$$

Now, let's do the integral.

$$\begin{aligned}\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx &= \int -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2}\,dx \\ &= -\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + c\end{aligned}$$

***Example 3*** Evaluate the following integral.

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx$$

***Solution***

This time the denominator is already factored so let's just jump right to the partial fraction decomposition.

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

Setting numerators gives,

$$x^2-29x+5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$$

In this case we aren't going to be able to just pick values of $x$ that will give us all the constants. Therefore, we will need to work this the second (and often longer) way. The first step is to multiply out the right side and collect all the like terms together. Doing this gives,

$$x^2-29x+5 = (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12A + 3B + 16D$$

In other words we will need to set the coefficients of like powers of $x$ equal. This will give a system of equations that can be solved.

$$\left.\begin{array}{ll} x^3: & A+C=0 \\ x^2: & -4A+B-8C+D=1 \\ x^1: & 3A+16C-8D=-29 \\ x^0: & -12A+3B+16D=5 \end{array}\right\} \Rightarrow \quad A=1, B=-5, C=-1, D=2$$

Now, let's take a look at the integral.

$$\begin{aligned} \int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} + \frac{-x+2}{x^2+3}\,dx \\ &= \int \frac{1}{x-4} - \frac{5}{(x-4)^2} - \frac{x}{x^2+3} + \frac{2}{x^2+3}\,dx \\ &= \ln|x-4| + \frac{5}{x-4} - \frac{1}{2}\ln|x^2+3| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c \end{aligned}$$

***Example 4*** Evaluate the following integral.

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}dx$$

***Solution***

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned}x^3+10x^2+3x+36 &= A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)\\ &=(A+B)x^4+(C-B)x^3+(8A+4B-C+D)x^2+\\ &\quad (-4B+4C-D+E)x+16A-4C-E\end{aligned}$$

Setting coefficient equal gives the following system.

$$\left.\begin{aligned} x^4: &\quad A+B=0\\ x^3: &\quad C-B=1\\ x^2: &\quad 8A+4B-C+D=10\\ x^1: &\quad -4B+4C-D+E=3\\ x^0: &\quad 16A-4C-E=36\end{aligned}\right\} \Rightarrow \quad A=2, B=-2, C=-1, D=1, E=0$$

Here's the integral.

$$\begin{aligned}\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}dx &= \int \frac{2}{x-1}+\frac{-2x-1}{x^2+4}+\frac{x}{(x^2+4)^2}dx\\ &=\int \frac{2}{x-1}-\frac{2x}{x^2+4}-\frac{1}{x^2+4}+\frac{x}{(x^2+4)^2}dx\\ &=2\ln|x-1|-\ln|x^2+4|-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{2}\frac{1}{x^2+4}+c\end{aligned}$$

If a rational function $\frac{R(x)}{Q(x)}$ is such that the degree of $R(x)$ is **greater than** the degree of $Q(x)$, then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{P(x)}{Q(x)}$$

where now $P(x)$ is a remainder term with the degree of $P(x)$ **less than** the degree of $Q(x)$ and our object is to integrate each term of the above representation.

To this point we've only looked at rational expressions where the degree of the numerator was strictly less that the degree of the denominator. Of course not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case.

***Example 5*** Evaluate the following integral.

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx$$

***Solution***

So, in this case the degree of the numerator is 4 and the degree of the denominator is 3. Therefore, partial fractions can't be done on this rational expression.

To fix this up we'll need to do long division on this to get it into a form that we can deal with. Here is the work for that.

$$\begin{array}{r} x-2 \\ x^3-3x^2 \overline{\big)\, x^4-5x^3+6x^2-18} \\ \underline{-\left(x^4-3x^3\right)} \\ -2x^3+6x^2-18 \\ \underline{-\left(-2x^3+6x^2\right)} \\ -18 \end{array}$$

$$\frac{x^4-5x^3+6x^2-18}{x^3-3x^2}=x-2-\frac{18}{x^3-3x^2}$$

and the integral becomes,

$$\int\frac{x^4-5x^3+6x^2-18}{x^3-3x^2}dx=\int x-2-\frac{18}{x^3-3x^2}dx$$
$$=\int x-2\,dx-\int\frac{18}{x^3-3x^2}dx$$

The first integral we can do easily enough and the second integral is now in a form that allows us to do partial fractions. So, let's get the general form of the partial fractions for the second integrand.

$$\frac{18}{x^2(x-3)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}$$

Setting numerators equal gives us,

$$18=Ax(x-3)+B(x-3)+Cx^2$$

| | | | |
|---|---|---|---|
| $x=0$ | $18=B(-3)$ | $\Rightarrow$ | $B=-6$ |
| $x=3$ | $18=C(9)$ | $\Rightarrow$ | $C=2$ |
| $x=1$ | $18=A(-2)+B(-2)+C=-2A+14$ | $\Rightarrow$ | $A=-2$ |

The integral is then,

$$\int\frac{x^4-5x^3+6x^2-18}{x^3-3x^2}dx=\int x-2\,dx-\int -\frac{2}{x}-\frac{6}{x^2}+\frac{2}{x-3}dx$$
$$=\frac{1}{2}x^2-2x+2\ln|x|-\frac{6}{x}-2\ln|x-3|+c$$

## 1.4 Integrals Involving Roots.

***Example 1*** Evaluate the following integral.

$$\int \frac{x+2}{\sqrt[3]{x-3}}\,dx$$

***Solution***

Sometimes when faced with an integral that contains a root we can use the following substitution to simplify the integral into a form that can be easily worked with.

$$u = \sqrt[3]{x-3}$$

So, instead of letting $u$ be the stuff under the radical as we often did in Calculus I we let $u$ be the whole radical. Now, there will be a little more work here since we will also need to know what $x$ is so we can substitute in for that in the numerator and so we can compute the differential, $dx$. This is easy enough to get however. Just solve the substitution for $x$ as follows,

$$x = u^3 + 3 \qquad\qquad dx = 3u^2\,du$$

Using this substitution the integral is now,

$$\int \frac{\left(u^3+3\right)+2}{u} 3u^2 du = \int 3u^4 + 15u\,du$$

$$= \frac{3}{5}u^5 + \frac{15}{2}u^2 + c$$

$$= \frac{3}{5}(x-3)^{\frac{5}{3}} + \frac{15}{2}(x-3)^{\frac{2}{3}} + c$$

So, sometimes, when an integral contains the root $\sqrt[n]{g(x)}$ the substitution,

$$u = \sqrt[n]{g(x)}$$

can be used to simplify the integral into a form that we can deal with.

***Example 2*** Evaluate the following integral.

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx$$

***Solution***

We'll do the same thing we did in the previous example. Here's the substitution and the extra work we'll need to do to get $x$ in terms of $u$.

$$u = \sqrt{x+10} \qquad x = u^2 - 10 \qquad dx = 2u\,du$$

With this substitution the integral is,

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx = \int \frac{2}{u^2 - 10 - 3u}(2u)\,du = \int \frac{4u}{u^2 - 3u - 10}\,du$$

This integral can now be done with partial fractions.

$$\frac{4u}{(u-5)(u+2)} = \frac{A}{u-5} + \frac{B}{u+2}$$

Setting numerators equal gives,

$$4u = A(u+2) + B(u-5)$$

Picking value of $u$ gives the coefficients.

$$u = -2 \qquad -8 = B(-7) \qquad B = \frac{8}{7}$$

$$u = 5 \qquad 20 = A(7) \qquad A = \frac{20}{7}$$

The integral is then,

$$\begin{aligned}\int \frac{2}{x - 3\sqrt{x+10}}\,dx &= \int \frac{\frac{20}{7}}{u-5} + \frac{\frac{8}{7}}{u+2}\,du \\ &= \frac{20}{7}\ln|u-5| + \frac{8}{7}\ln|u+2| + c \\ &= \frac{20}{7}\ln\left|\sqrt{x+10} - 5\right| + \frac{8}{7}\ln\left|\sqrt{x+10} + 2\right| + c\end{aligned}$$

# *Sheet No. 1 Problems*

## *(A)*

### *Integration by Parts*

Evaluate each of the following integrals.

1. $\int 4x\cos(2-3x)\,dx$

2. $\int_{6}^{0}(2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx$

3. $\int (3t+t^2)\sin(2t)\,dt$

4. $\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$

5. $\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)dz$

6. $\int_{0}^{\pi} x^2\cos(4x)\,dx$

7. $\int t^7\sin(2t^4)\,dt$

8. $\int y^6\cos(3y)\,dy$

9. $\int (4x^3-9x^2+7x+3)\mathbf{e}^{-x}\,dx$

# Sheet No. 1 Problems

**(B)**

## Integrals Involving Trig Functions

Evaluate each of the following integrals.

1. $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$

2. $\int \sin^8(3z)\cos^5(3z)\,dz$

3. $\int \cos^4(2t)\,dt$

4. $\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)dw$

5. $\int \sec^6(3y)\tan^2(3y)\,dy$

6. $\int \tan^3(6x)\sec^{10}(6x)\,dx$

7. $\int_0^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz$

8. $\int \cos(3t)\sin(8t)\,dt$

# Sheet No. 1 Problems

## (C)

### Partial Fractions

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$

2. $\displaystyle\int \frac{8-3t}{10t^2+13t-3}\,dt$

3. $\displaystyle\int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$

4. $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$

5. $\displaystyle\int_{2}^{4} \frac{3z^2+1}{(z+1)(z-5)^2}\,dz$

6. $\displaystyle\int \frac{4x-11}{x^3-9x^2}\,dx$

7. $\displaystyle\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz$

8. $\displaystyle\int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,dt$

# Sheet No. 1 Problems

**(D)**

## Integrals Involving Roots

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$

2. $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$

3. $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$