Example 8 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^{\infty} \frac{1}{x^2}\,dx$$

Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_0^{\infty} \frac{1}{x^2}\,dx = \int_0^{1} \frac{1}{x^2}\,dx + \int_1^{\infty} \frac{1}{x^2}\,dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\begin{aligned}\int_0^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^+} \int_t^{1} \frac{1}{x^2}\,dx \\ &= \lim_{t\to 0^+} \left(-\frac{1}{x} \right)\Bigg|_t^1 \\ &= \lim_{t\to 0^+} \left(-1 + \frac{1}{t} \right) \\ &= \infty\end{aligned}$$

2.2 <u>*Comparison Test for Improper Integrals*</u>.

Now that we've seen how to actually compute improper integrals we need to address one more topic about them. Often we aren't concerned with the actual value of these integrals. Instead we might only be interested in whether the integral is convergent or divergent. Also, there will be some integrals that we simply won't be able to integrate and yet we would still like to know if they converge or diverge.

To deal with this we've got a test for convergence or divergence that we can use to help us answer the question of convergence for an improper integral.

Comparison Test

If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ converges then so does $\int_a^\infty g(x)\,dx$.
2. If $\int_a^\infty g(x)\,dx$ diverges then so does $\int_a^\infty f(x)\,dx$.

Note that if you think in terms of area the Comparison Test makes a lot of sense. If $f(x)$ is larger than $g(x)$ then the area under $f(x)$ must also be larger than the area under $g(x)$.

So, if the area under the larger function is finite (*i.e.* $\int_a^\infty f(x)\,dx$ converges) then the area under the smaller function must also be finite (*i.e.* $\int_a^\infty g(x)\,dx$ converges). Likewise, if the area under the smaller function is infinite (*i.e.* $\int_a^\infty g(x)\,dx$ diverges) then the area under the larger function must also be infinite (*i.e.* $\int_a^\infty f(x)\,dx$ diverges).

Example 1 Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$$

Solution.

So, it seems like it would be nice to have some idea as to whether the integral converges or diverges ahead of time so we will know whether we will need to look for a larger (and convergent) function or a smaller (and divergent) function.

Therefore, it seems likely that the denominator will determine the convergence/divergence of this integral and we know that

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges since $p = 2 > 1$ by the fact in the previous section. So let's guess that this integral will converge.

So we now know that we need to find a function that is larger than

$$\frac{\cos^2 x}{x^2}$$

and also converges. Making a fraction larger is actually a fairly simple process. We can either make the numerator larger or we can make the denominator smaller. In this case we can't do a lot about the denominator. However we can use the fact that $0 \le \cos^2 x \le 1$ to make the numerator larger (*i.e.* we'll replace the cosine with something we know to be larger, namely 1). So,

$$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

Now, as we've already noted

$$\int_2^{\infty} \frac{1}{x^2}\,dx$$

converges and so by the Comparison Test we know that

$$\int_2^{\infty} \frac{\cos^2 x}{x^2}\,dx$$

must also converge.

Example 2 Determine if the following integral is convergent or divergent.

$$\int_3^{\infty} \frac{1}{x + \mathbf{e}^x}\,dx$$

Solution

Let's first take a guess about the convergence of this integral. As noted after the fact in the last section about

The question then is which one to drop? Let's first drop the exponential. Doing this gives,

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{x}$$

This is a problem however, since

$$\int_3^{\infty} \frac{1}{x}\,dx$$

diverges by the fact. We've got a larger function that is divergent. This doesn't say anything about the smaller function. Therefore, we chose the wrong one to drop.

Let's try it again and this time let's drop the x.

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

Also,

$$\begin{aligned}\int_3^\infty \mathbf{e}^{-x}\,dx &= \lim_{t\to\infty}\int_3^t \mathbf{e}^{-x}\,dx\\ &= \lim_{t\to\infty}\left(-\mathbf{e}^{-t}+\mathbf{e}^{-3}\right)\\ &= \mathbf{e}^{-3}\end{aligned}$$

So, $\int_3^\infty \mathbf{e}^{-x}\,dx$ is convergent. Therefore, by the Comparison test

$$\int_3^\infty \frac{1}{x+\mathbf{e}^{x}}\,dx$$

is also convergent.

Example 3 Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

Solution

This is where the second change will come into play. As before we know that both x and the exponential are positive. However, this time since we are subtracting the exponential from the x if we were to drop the exponential the denominator will become larger and so the fraction will become smaller. In other words,

$$\frac{1}{x-\mathbf{e}^{-x}} > \frac{1}{x}$$

and we know that

$$\int_3^\infty \frac{1}{x}\,dx$$

diverges and so by the Comparison Test we know that

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

must also diverge.

Example 4 Determine if the following integral is convergent or divergent.

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

Solution

Therefore, since the exponent on the denominator is less than 1 we can guess that the integral will probably diverge. We will need a smaller function that also diverges.

We know that $0 \le \sin^4(2x) \le 1$. In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1+3\sin^4(2x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$$

and

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges so by the Comparison Test

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

also diverges.

Problems: Sheet No. 2

Improper Integrals

Determine if each of the following integrals converge or diverge. If the integral converges determine its value.

1. $\int_0^\infty (1+2x)\mathbf{e}^{-x}\,dx$

2. $\int_{-\infty}^0 (1+2x)\mathbf{e}^{-x}\,dx$

3. $\int_{-5}^1 \frac{1}{10+2z}\,dz$

4. $\int_1^2 \frac{4w}{\sqrt[3]{w^2-4}}\,dw$

5. $\int_{-\infty}^1 \sqrt{6-y}\,dy$

6. $\int_2^\infty \frac{9}{(1-3z)^4}\,dz$

7. $\int_0^4 \frac{x}{x^2-9}\,dx$

8. $\int_{-\infty}^\infty \frac{6w^3}{\left(w^4+1\right)^2}\,dw$

9. $\int_1^4 \frac{1}{x^2+x-6}\,dx$

10. $\int_{-\infty}^0 \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$

Problems: Sheet No. 2

Comparison Test for Improper Integrals

Use the Comparison Test to determine if the following integrals converge or diverge.

1. $\int_{1}^{\infty} \frac{1}{x^3+1} dx$

2. $\int_{3}^{\infty} \frac{z^2}{z^3-1} dz$

3. $\int_{4}^{\infty} \frac{e^{-y}}{y} dy$

4. $\int_{1}^{\infty} \frac{z-1}{z^4+2z^2} dz$