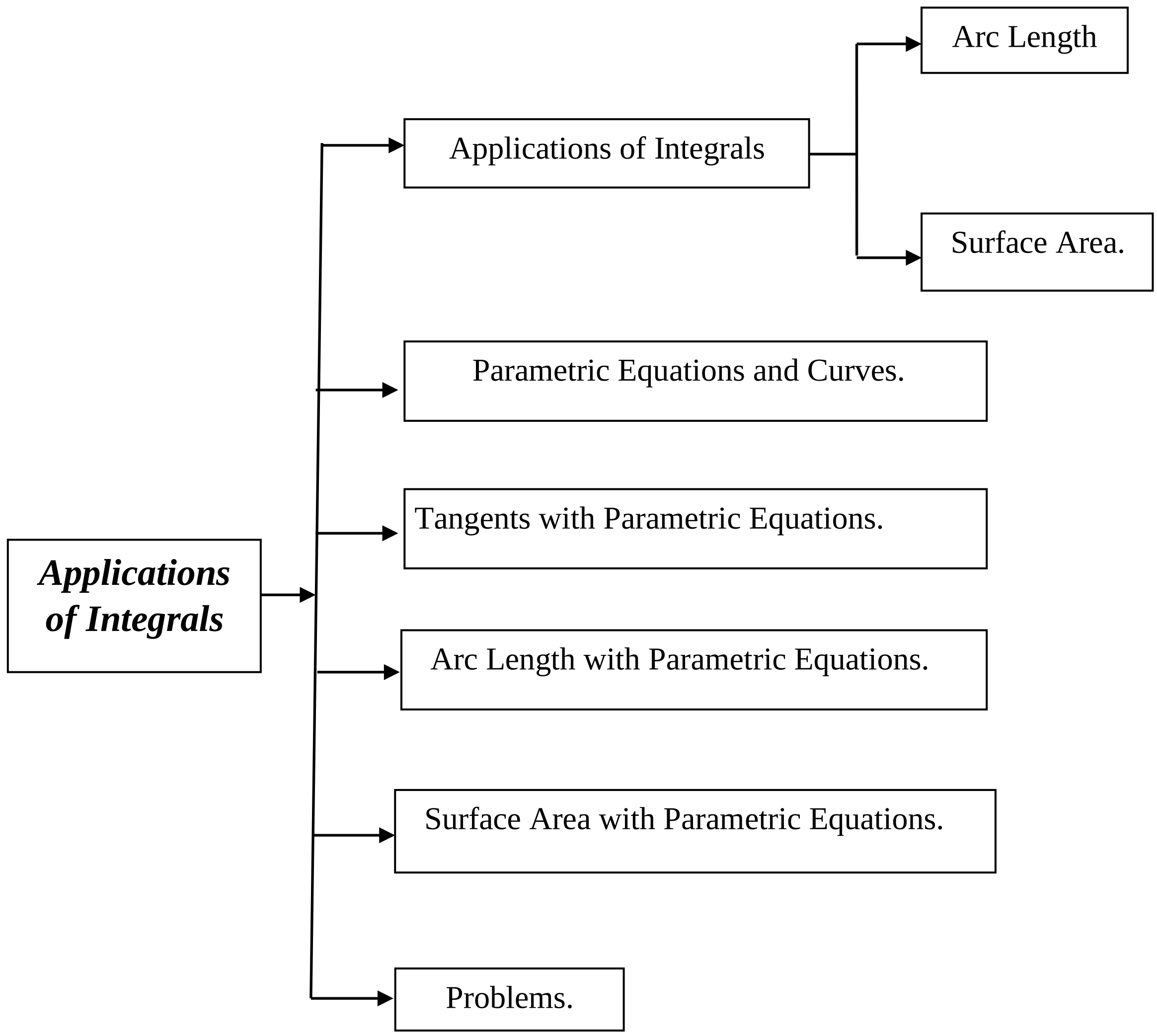
Lecture – Three

Applications of Integrals

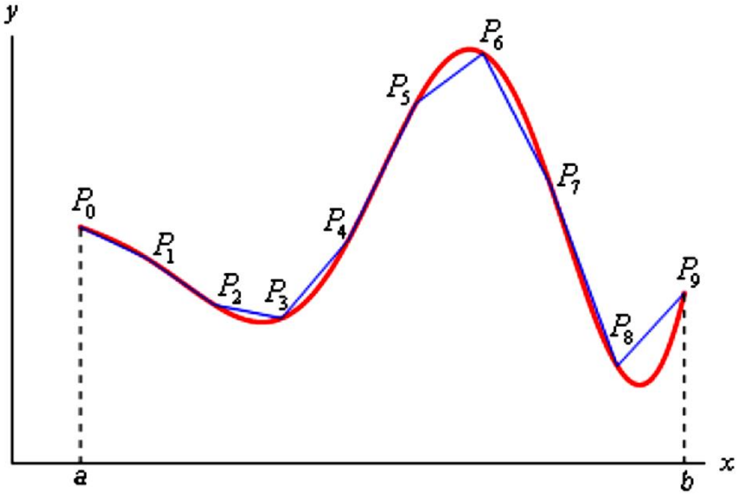


3.1 Applications of Integrals.

A- Arc Length.

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a, b]$. We'll also need to assume that the derivative is continuous on $[a, b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

$$\Delta y_i = f'(x_i^*)\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

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The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left| P_{i-1}\ P_i \right|$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\left[f'\left(x_i^*\right)\right]^2}\ \ \Delta x$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\, dx$$

A slightly more convenient notation (in my opinion anyway) is the following.

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c,d]$. This formula is,

$$L = \int_c^d \sqrt{1+\left[h'(y)\right]^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Example 1 Determine the length of $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$.

Solution

In this case we'll need to use the first *ds* since the function is in the form $y = f(x)$. So, let's get the derivative out of the way.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \qquad \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Let's also get the root out of the way since there is often simplification that can be done and there's no reason to do that inside the integral.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec x \, dx \\ &= \ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{4}} \\ &= \ln\left(\sqrt{2} + 1\right) \end{aligned}$$

Example 2 Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \le y \le 4$.

Solution

Let's compute the derivative and the root.

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y-1} = \sqrt{y}$$

As you can see keeping the function in the form $x = h(y)$ is going to lead to a very easy integral. To see what would happen if we tried to work with the function in the form $y = f(x)$ see the next example.

Let's get the length.

$$\begin{aligned} L &= \int_1^4 \sqrt{y}\, dy \\ &= \frac{2}{3} y^{\frac{3}{2}} \Big|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

Example 3 Redo the previous example using the function in the form $y = f(x)$ instead.

Solution

In this case the function and its derivative would be,

$$y = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad \frac{dy}{dx} = \left(\frac{3x}{2}\right)^{-\frac{1}{3}}$$

The root in the arc length formula would then be.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \sqrt{\frac{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}$$

All the simplification work above was just to put the root into a form that will allow us to do the integral.

Now, before we write down the integral we'll also need to determine the limits. This particular ds requires x limits of integration and we've got y limits. They are easy enough to get however. Since we know x as a function of y all we need to do is plug in the original y limits of integration and get the x limits of integration. Doing this gives,

$$0 \le x \le \frac{2}{3}(3)^{\frac{3}{2}}$$

Not easy limits to deal with, but there they are.

Let's now write down the integral that will give the length.

$$L = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}} dx$$

That's a really unpleasant looking integral. It can be evaluated however using the following substitution.

$$u = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad du = \left(\frac{3x}{2}\right)^{-\frac{1}{3}} dx$$

$$x = 0 \quad \Rightarrow \quad u = 1$$

$$x = \frac{2}{3}(3)^{\frac{3}{2}} \quad \Rightarrow \quad u = 4$$

Using this substitution the integral becomes,

$$L = \int_1^4 \sqrt{u}\, du = \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^4 = \frac{14}{3}$$

So, we got the same answer as in the previous example. Although that shouldn't really be all that surprising since we were dealing with the same curve.

Example 4 Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that y is positive.

Solution

We'll use the second ds for this one as the function is already in the correct form for that one. Also, the other ds would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y^2}$$

Before writing down the length notice that we were given x limits and we will need y limits for this ds. With the assumption that y is positive these are easy enough to get. All we need to do is plug x into our equation and solve for y. Doing this gives,

$$0 \le y \le 1$$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1+y^2}\, dy$$

This integral will require the following trig substitution.

$$y = \tan\theta \qquad\qquad dy = \sec^2\theta\, d\theta$$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right| = \sec\theta$$

The length is then,

$$L = \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)$$