B- ***Surface Area***.

The surface area of a frustum is given by,

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad r_1 = \text{radius of right end}$$

$$r_2 = \text{radius of left end}$$

and l is the length of the slant of the frustum.

For the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$r_1 = f(x_i)$$

$$r_2 = f(x_{i-1})$$

$$l = |P_{i-1}\ P_i| \quad (\text{length of the line segment connecting } P_i \text{ and } P_{i-1})$$

We know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is "small" and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)|P_{i-1}\ P_i|$$
$$\approx 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \Delta x$$

and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \Delta x$$
$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c,d]$ about the y-axis. This would give the following formula.

$$S = \int_c^d 2\pi h(y)\sqrt{1+\left[h'(y)\right]^2}\,dy$$

Surface Area Formulas

$$S = \int 2\pi y\,ds \qquad \text{rotation about } x-\text{axis}$$
$$S = \int 2\pi x\,ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f(x),\ a \le x \le b$$
$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Example 1 Determine the surface area of the solid obtained by rotating $y = \sqrt{9-x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution

The formula that we'll be using here is,

$$S = \int 2\pi y\,ds$$

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx}=\frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}}(-2x)=-\frac{x}{\left(9-x^2\right)^{\frac{1}{2}}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{x^2}{9-x^2}}=\sqrt{\frac{9}{9-x^2}}=\frac{3}{\sqrt{9-x^2}}$$

Here's the integral for the surface area,

$$S=\int_{-2}^{2}2\pi y\frac{3}{\sqrt{9-x^2}}\,dx$$

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

The surface area is then,

$$\begin{aligned}S&=\int_{-2}^{2}2\pi\sqrt{9-x^2}\frac{3}{\sqrt{9-x^2}}\,dx\\&=\int_{-2}^{2}6\pi\,dx\\&=24\pi\end{aligned}$$

Example 2 Determine the surface area of the solid obtained by rotating $y=\sqrt[3]{x}$, $1\le y\le 2$ about the y-axis. Use both ds's to compute the surface area.

Solution

Note that we've been given the function set up for the first ds and limits that work for the second ds.

Solution 1

This solution will use the first ds listed above. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}}+1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \le x \le 8$. The integral for the surface area is then,

$$S = \int_1^8 2\pi x \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}\,dx$$

$$= \frac{2\pi}{3}\int_1^8 x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\,dx$$

Using the substitution

$$u = 9x^{\frac{4}{3}}+1 \qquad\qquad du = 12x^{\frac{1}{3}}\,dx$$

the integral becomes,

$$S = \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du$$

$$= \frac{\pi}{27}u^{\frac{3}{2}}\Big|_{10}^{145}$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Solution 2

This time we'll use the second ds. So, we'll first need to solve the equation for x. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad\qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x\sqrt{1+9y^4}\,dy$$

We used the original y limits this time because we picked up a dy from the ds. Also note that the presence of the dy means that this time, unlike the first solution, we'll need to substitute in for the x. Doing that gives,

$$\begin{aligned} S &= \int_1^2 2\pi y^3\sqrt{1+9y^4}\,dy \qquad\qquad u = 1+9y^4 \\ &= \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du \\ &= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48 \end{aligned}$$

Note that after the substitution the integral was identical to the first solution and so the work was skipped.

3.2 *Parametric Equations and Curves.*

To this point (in both Calculus I and Calculus II) we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form.

Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \quad (\text{top}) \qquad\qquad x = \sqrt{r^2 - y^2} \quad (\text{right side})$$

$$y = -\sqrt{r^2 - x^2} \quad (\text{bottom}) \qquad\qquad x = -\sqrt{r^2 - y^2} \quad (\text{left side})$$

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining y in terms of x ($y = f(x)$) or x in terms of y ($x = h(y)$) we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad\qquad y = g(t)$$

Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve**.

Example 1 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad\qquad y = 2t - 1$$

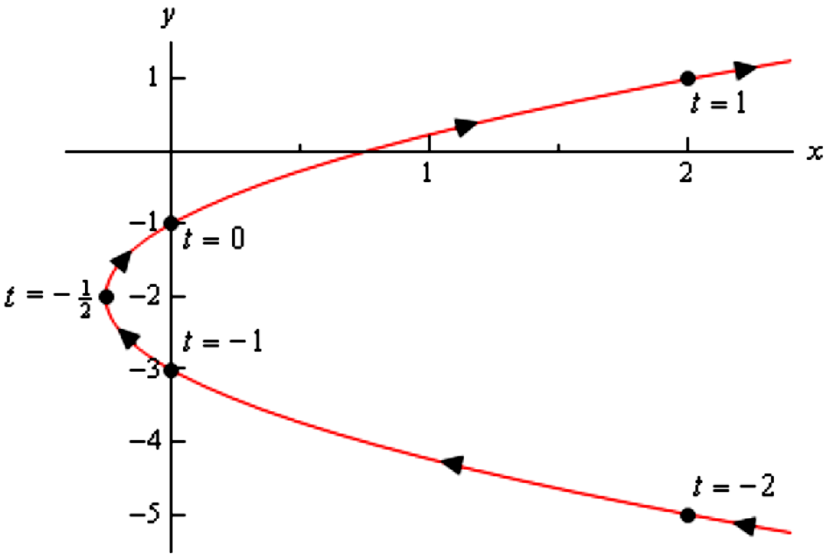
Solution

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

We have one more idea to discuss before we actually sketch the curve. Parametric curves have a **direction of motion**. The direction of motion is given by increasing t. So, when plotting parametric curves we also include arrows that show the direction of motion.

Here is the sketch of this parametric curve.

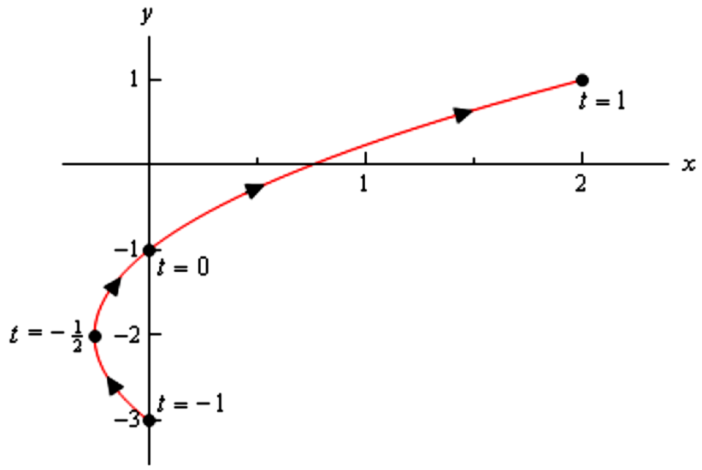


Example 2 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \le t \le 1$$

Solution

Note that the only difference here is the presence of the limits on *t*. All these limits do is tell us that we can't take any value of *t* outside of this range. Therefore, the parametric curve will only be a portion of the curve above. Here is the parametric curve for this example.



Example 3 Sketch the parametric curve for the following set of parametric equations. Clearly indicate direction of motion.

$$x = 5\cos t \qquad y = 2\sin t \qquad 0 \le t \le 2\pi$$

Solution

An alternate method that we could have used here was to solve the two parametric equations for sine and cosine as follows,

$$\cos t = \frac{x}{5} \qquad \sin t = \frac{y}{2}$$

Then, recall the trig identity we used above and these new equation we get,

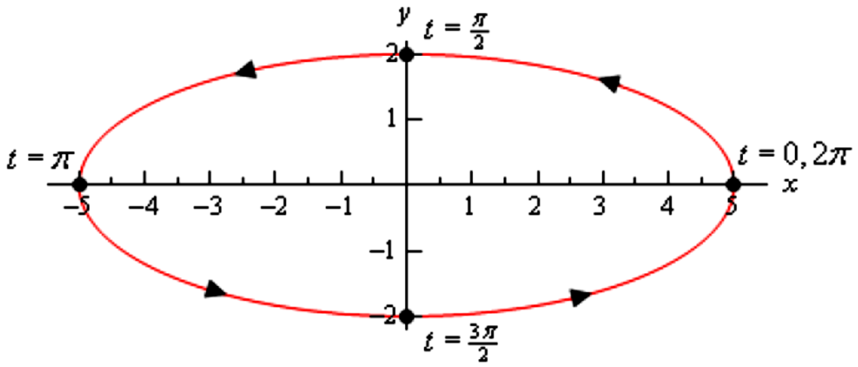
$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{25} + \frac{y^2}{4}$$

So, here is a table of values for this set of parametric equations.

t	x	y
0	5	0
$\frac{\pi}{2}$	0	2
π	-5	0
$\frac{3\pi}{2}$	0	-2
2π	5	0

It looks like we are moving in a counter-clockwise direction about the ellipse and it also looks like we'll make exactly one complete trace of the ellipse in the range given.

Here is a sketch of the parametric curve.



Example 4

The path of a particle is given by the following set of parametric equations.

$$x = 3\cos(2t) \qquad y = 1 + \cos^2(2t)$$

Completely describe the path of this particle. Do this by sketching the path, determining limits on x and y and giving a range of t's for which the path will be traced out exactly once (provide it traces out more than once of course).

Solution

Eliminating the parameter this time will be a little different. We only have cosines this time and we'll use that to our advantage. We can solve the x equation for cosine and plug that into the equation for y. This gives,

$$\cos(2t) = \frac{x}{3} \qquad\qquad y = 1 + \left(\frac{x}{3}\right)^2 = 1 + \frac{x^2}{9}$$

This time we've got a parabola that opens upward. We also have the following limits on x and y.

$$-1 \le \cos(2t) \le 1 \qquad -3 \le 3\cos(2t) \le 3 \qquad -3 \le x \le 3$$

$$0 \le \cos^2(2t) \le 1 \qquad 1 \le 1 + \cos^2(2t) \le 2 \qquad 1 \le y \le 2$$

So, again we only trace out a portion of the curve. Here's a set of evaluations so we can determine a range of t's for one trace of the curve.

t	x	y
0	3	2
$\frac{\pi}{4}$	0	1
$\frac{\pi}{2}$	-3	2
$\frac{3\pi}{4}$	0	1
π	3	2

So, it looks like the particle, again, will continuously trace out this portion of the curve and will make one trace in the range $0 \le t \le \frac{\pi}{2}$. Here is a sketch of the particle's path with a few value of t on it.