3.3 Tangents with Parametric Equations.

In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x-a), \quad \text{where} \quad m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the x and y coordinates of the point.

So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for x and y. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F(f(t))$$

Now, differentiate with respect to t and notice that we'll need to use the Chain Rule on the right hand side.

$$g'(t) = F'(f(t))\, f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to t while derivatives of upper case functions are with respect to x. So, to make sure that we keep this straight let's rewrite things as follows.

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\frac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Derivative for Parametric Equations

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Example 1 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad y = t^2$$

at (0,4).

Solution

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of t and all we've got is an x-y coordinate pair. The next step then is to determine the value(s) of t which will give this point. We find these by plugging the x and y values into the parametric equations and solving for t.

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

$t = -2$

Since we already know the x and y-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

$t = 2$

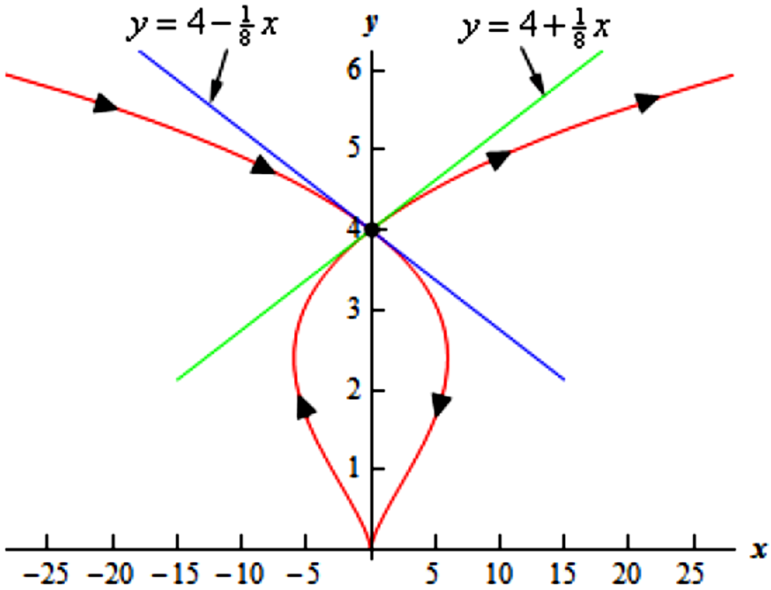
Again, all we need is the slope.

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

A quick graph of the parametric curve will explain what is going on here.



Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt} = 0, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have,

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt} = 0, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Example 2 Determine the x-y coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t \qquad\qquad y = 3t^2 - 9$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 = 3\left(t^2 - 1\right) \qquad\qquad \frac{dy}{dt} = 6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t = 0 \qquad \Rightarrow \qquad t = 0$$

Now, this is the value of t which gives the horizontal tangents and we were asked to find the x-y coordinates of the point. To get these we just need to plug t into the parametric equations. Therefore, the only horizontal tangent will occur at the point (0,-9).

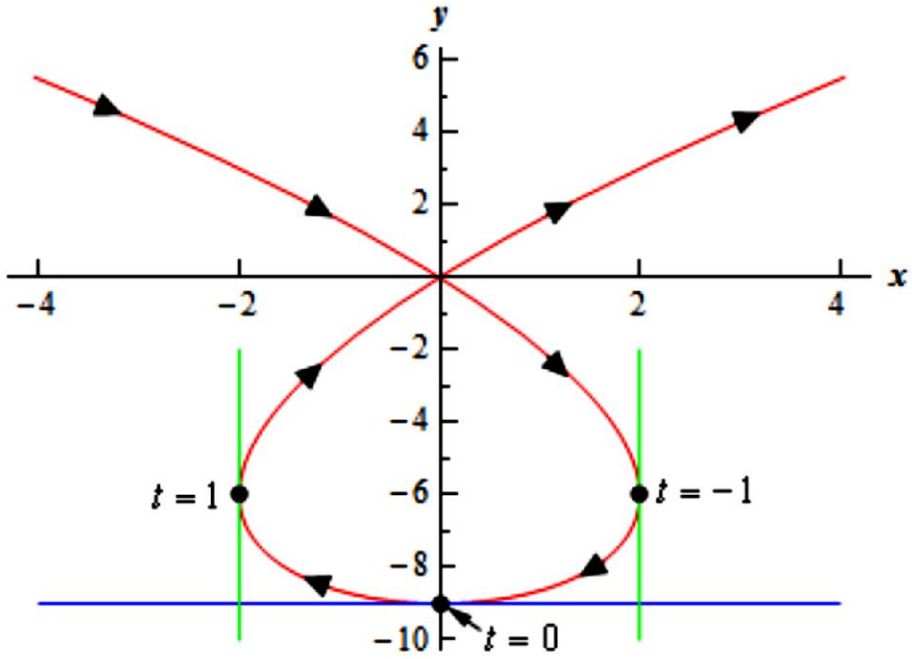
Vertical Tangents

In this case we need to solve,

$$3\left(t^2 - 1\right) = 0 \qquad \Rightarrow \qquad t = \pm 1$$

The two vertical tangents will occur at the points (2,-6) and (-2,-6).

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



3.4 Arc Length with Parametric Equations.

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad \text{for } \alpha \le t \le \beta$$

To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}dt$$

The arc length formula then becomes,

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}dt = \int_\alpha^\beta \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,\frac{dx}{dt}dt$$

$$L = \int_\alpha^\beta \frac{1}{\left|\frac{dx}{dt}\right|}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\frac{dx}{dt}dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

Arc Length for Parametric Equations

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Notice that we could have used the second formula for ds above if we had assumed instead that

$$\frac{dy}{dt} \ge 0 \qquad \text{for } \alpha \le t \le \beta$$

Example 1 Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad y = 3\cos(t) \qquad 0 \le t \le 2\pi$$

Solution

So, we can use the formula we derived above. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t) \qquad \frac{dy}{dt} = -3\sin(t)$$

The length is then,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)}\, dt \\ &= 3\int_0^{2\pi} dt \\ &= 6\pi \end{aligned}$$

Example 2 Use the arc length formula for the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le t \le 2\pi$$

Solution

Notice that this is the identical circle that we had in the previous example and so the length is still 6π. However, for the range given we know it will trace out the curve three times instead once as required for the formula. Despite that restriction let's use the formula anyway and see what happens.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(t) + 81\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 9\, dt \\ &= 18\pi \end{aligned}$$

The arc length formula can be summarized as,

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t),\ y = g(t),\ \alpha \le t \le \beta$$

3.5 Surface Area with Parametric Equations.

In this final section of looking at calculus applications with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the x or y-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y \, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x \, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t), y = g(t), \alpha \le t \le \beta$$

which is exactly what we need.

Example 1 Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = -3\cos^2\theta\sin\theta \qquad \frac{dy}{dt} = 3\sin^2\theta\cos\theta$$

Before plugging into the surface area formula let's get the ds out of the way.

$$\begin{aligned} ds &= \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\,dt \\ &= 3\left|\cos\theta\sin\theta\right|\sqrt{\cos^2\theta+\sin^2\theta} \\ &= 3\cos\theta\sin\theta \end{aligned}$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of θ given.

Now let's get the surface area and don't forget to also plug in for the y.

$$\begin{aligned} S &= \int 2\pi y\,ds \\ &= 2\pi\int_0^{\frac{\pi}{2}} \sin^3\theta\left(3\cos\theta\sin\theta\right)\,d\theta \\ &= 6\pi\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\,d\theta \qquad\qquad u=\sin\theta \\ &= 6\pi\int_0^1 u^4\,du \\ &= \frac{6\pi}{5} \end{aligned}$$

Problems: Sheet No. 3

A- Arc Length.

1. Set up, but do not evaluate, an integral for the length of $y=\sqrt{x+2}$, $1\le x\le 7$ using,

(a) $ds=\sqrt{1+\left[\dfrac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\dfrac{dx}{dy}\right]^2}\,dy$

2. Set up, but do not evaluate, an integral for the length of $x=\cos(y)$, $0\le x\le \frac{1}{2}$ using,

(a) $ds=\sqrt{1+\left[\dfrac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\dfrac{dx}{dy}\right]^2}\,dy$

3. Determine the length of $y=7(6+x)^{\frac{3}{2}}$, $189\le y\le 875$.

4. Determine the length of $x=4(3+y)^2$, $1\le y\le 4$.

B- Surface Area.

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $x=\sqrt{y+5}$, $\sqrt{5}\le x\le 3$ about the y-axis using,

(a) $ds=\sqrt{1+\left[\dfrac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\dfrac{dx}{dy}\right]^2}\,dy$

2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y=\sin(2x)$, $0\le x\le \frac{\pi}{8}$ about the x-axis using,

(a) $ds=\sqrt{1+\left[\dfrac{dy}{dx}\right]^2}\,dx$

Problems: Sheet No. 3

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = x^3 + 4$, $1 \le x \le 5$ about the given axis. You can use either ds.

   (a) x-axis

   (b) y-axis

4. Find the surface area of the object obtained by rotating $y = 4 + 3x^2$, $1 \le x \le 2$ about the y-axis.

5. Find the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis.

C- Parametric Equations and Curves.

For problems 1 – 6 eliminate the parameter for the given set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

1. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2$

2. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2 \qquad 0 \le t \le 3$

3. $x = \sqrt{t+1} \qquad y = \dfrac{1}{t+1} \qquad t > -1$

4. $x = 3\sin(t) \qquad y = -4\cos(t) \qquad 0 \le t \le 2\pi$

5. $x = 3\sin(2t) \qquad y = -4\cos(2t) \qquad 0 \le t \le 2\pi$

6. $x = 3\sin\left(\frac{1}{3}t\right) \qquad y = -4\cos\left(\frac{1}{3}t\right) \qquad 0 \le t \le 2\pi$

For problems 7 – 11 the path of a particle is given by the set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on x and y.

(*iii*) A range of t's for a single trace of the parametric curve.

Problems: Sheet No. 3

7. $x = 3 - 2\cos(3t) \qquad y = 1 + 4\sin(3t)$

8. $x = 4\sin\left(\frac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\frac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$

9. $x = \sqrt{4 + \cos\left(\frac{5}{2}t\right)} \qquad y = 1 + \frac{1}{3}\cos\left(\frac{5}{2}t\right) \qquad -48\pi \le t \le 2\pi$

10. $x = 2\mathbf{e}^{t} \qquad y = \cos\left(1 + \mathbf{e}^{3t}\right) \qquad 0 \le t \le \frac{3}{4}$

11. $x = \frac{1}{2}\mathbf{e}^{-3t} \qquad y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8$

D- Tangents with Parametric Equations.

For problems 1 and 2 compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the given set of parametric equations.

1. $x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$

2. $x = \mathbf{e}^{-7t} + 2 \qquad y = 6\mathbf{e}^{2t} + \mathbf{e}^{-3t} - 4t$

For problems 3 and 4 find the equation of the tangent line(s) to the given set of parametric equations at the given point.

3. $x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t)$ at $t = \frac{\pi}{2}$

4. $x = t^2 - 2t - 11 \qquad y = t(t-4)^3 - 3t^2(t-4)^2 + 7$ at $(-3, 7)$

5. Find the values of t that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x = t^5 - 7t^4 - 3t^3 \qquad y = 2\cos(3t) + 4t$$

E- Area with Parametric Equations.

For problems 1 and 2 determine the area of the region below the parametric curve given by the set of parametric equations. For each problem you may assume that each curve traces out exactly once from right to left for the given range of t. For these problems you should only use the given parametric equations to determine the answer.

1. $x = 4t^3 - t^2 \qquad y = t^4 + 2t^2 \qquad 1 \le t \le 3$

2. $x = 3 - \cos^3(t) \qquad y = 4 + \sin(t) \qquad 0 \le t \le \pi$

Problems: Sheet No. 3

F- Arc Length with Parametric Equations.

For problems 1 and 2 determine the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. $x = 8t^{\frac{3}{2}} \qquad y = 3 + (8-t)^{\frac{3}{2}} \qquad 0 \le t \le 4$

2. $x = 3t + 1 \qquad y = 4 - t^2 \qquad -2 \le t \le 0$

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

For problems 4 and 5 set up, but do not evaluate, an integral that gives the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. $x = 2 + t^2 \qquad y = \mathbf{e}^t \sin(2t) \qquad 0 \le t \le 3$

5. $x = \cos^3(2t) \qquad y = \sin\left(1 - t^2\right) \qquad -\tfrac{3}{2} \le t \le 0$

G- Surface Area with Parametric Equations.

For problems1 – 3 determine the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. Rotate $x = 3 + 2t \qquad y = 9 - 3t \qquad 1 \le t \le 4$ about the y-axis.

2. Rotate $x = 9 + 2t^2 \qquad y = 4t \qquad 0 \le t \le 2$ about the x-axis.

3. Rotate $x = 3\cos(\pi t) \qquad y = 5t + 2 \qquad 0 \le t \le \tfrac{1}{2}$ about the y-axis.

For problems 4 and 5 set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. Rotate $x = 1 + \ln\left(5 + t^2\right) \qquad y = 2t - 2t^2 \qquad 0 \le t \le 2$ about the x-axis.

5. Rotate $x = 1 + 3t^2 \qquad y = \sin(2t)\cos\left(\tfrac{1}{4}t\right) \qquad 0 \le t \le \tfrac{1}{2}$ about the y-axis.