4.3 Tangents with Polar Coordinates.

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \qquad\qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$= \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad\qquad = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example 1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \dfrac{\pi}{6}$.

Solution
We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\dfrac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3+8\sin\theta)\cos\theta}{8\cos^2\theta - (3+8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

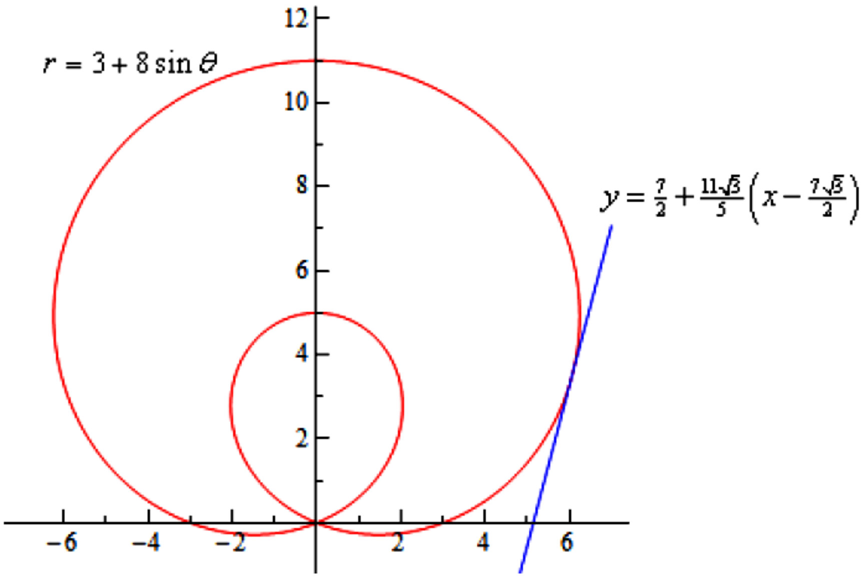
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



4.4 Arc Length with Polar Coordinates.

In this section we'll look at the arc length of the curve given by,

$$r = f(\theta) \qquad \alpha \le \theta \le \beta$$

where we also assume that the curve is traced out exactly once. Just as we did with the tangent lines in polar coordinates we'll first write the curve in terms of a set of parametric equations,

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= f(\theta)\cos\theta & &= f(\theta)\sin\theta \end{aligned}$$

and we can now use the parametric formula for finding the arc length.

We'll need the following derivatives for these computations.

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta & \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta & &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{aligned}$$

We'll need the following for our ds.

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2 \end{aligned}$$

The arc length formula for polar coordinates is then,

$$L = \int ds$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 1 Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

Solution

Okay, let's just jump straight into the formula since this is a fairly simple function.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$

We'll need to use a trig substitution here.

$$\theta = \tan x \qquad\qquad d\theta = \sec^2 x\, dx$$

$$\theta = 0 \qquad 0 = \tan x \qquad x = 0$$

$$\theta = 1 \qquad 1 = \tan x \qquad x = \frac{\pi}{4}$$

$$\sqrt{\theta^2 + 1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = \left|\sec x\right| = \sec x$$

The arc length is then,

$$\begin{aligned} L &= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^3 x\, dx \\ &= \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

4.5 Area Polar Coordinates.

The equation of a curve in polar coordinates is given by $r = f(\theta)$. To find the area bounded by the curve $r = f(\theta)$, the rays $\theta = \alpha$ and $\theta = \beta$, divide the angle $\beta - \alpha$ into n-parts by defining $\Delta\theta = \frac{\beta-\alpha}{n}$ and then defining the rays

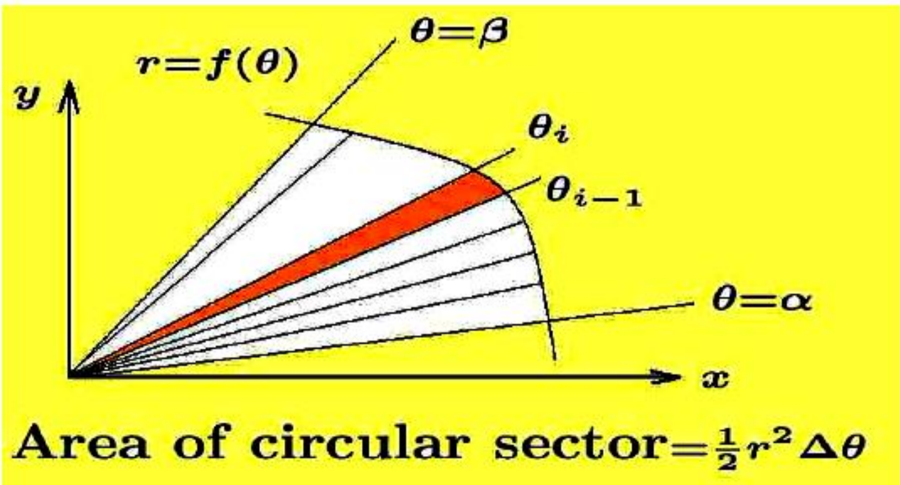
$$\theta_0 = \alpha,\ \theta_1 = \theta_0 + \Delta\theta, \dots, \theta_i = \theta_{i-1} + \Delta\theta, \dots, \theta_n = \theta_{n-1} + \Delta\theta = \beta$$

The area between the rays $\theta = \theta_{i-1}$, $\theta = \theta_i$ and the curve $r = f(\theta)$, illustrated in the figure below, is approximated by a circular sector with area element

$$dA_i = \frac{1}{2} r_i^2 \Delta\theta_i = \frac{1}{2} f^2(\theta_i)\,\Delta\theta_i$$

where $\Delta\theta_i = \theta_i - \theta_{i-1}$ and $r_i = f(\theta_i)$. A summation of these elements of area between the rays $\theta = \alpha$ and $\theta = \beta$ gives the approximate area

$$\sum_{i=1}^{n} dA_i = \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i = \sum_{i=1}^{n} \frac{1}{2} f^2(\theta_i)\,\Delta\theta_i$$



Approximation of area by summation of circular sectors.

This approximation gets better as $\Delta\theta_i$ gets smaller. Using the fundamental theorem of integral calculus, it can be shown that in the limit as $n \to \infty$, the equation defines the element of area $dA = \frac{1}{2} r^2\, d\theta$. A summation of these elements of area gives

$$\text{Polar Area} = \int_{\alpha}^{\beta} dA = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\, d\theta$$

Example 1. Find the area bounded by the polar curve

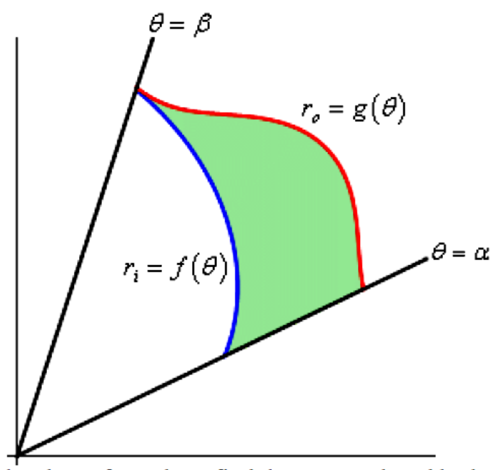
$$r = 2r_0 \cos\theta \quad \text{for } 0 \le \theta \le \pi.$$

Solution

One finds that the polar curve $r = 2r_0 \cos\theta$, for $0 \le \theta \le \pi$, is a circle of radius r_0 which has its center at the point $(r_0, 0)$ in polar coordinates. Using the area formula given by equation (3.121) one obtains

$$\text{Area} = \frac{1}{2}\int_0^{\pi} (2r_0\cos\theta)^2\,d\theta = 2r_0^2\int_0^{\pi}\cos^2\theta\,d\theta = r_0^2\int_0^{\pi}(\cos 2\theta + 1)\,d\theta = r_0^2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi} = \pi r_0^2$$

So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch were we want to find the area between two curves.

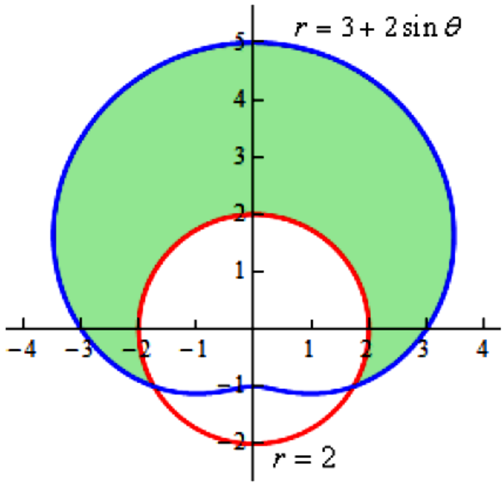


In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right)d\theta$$

Example 2 Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution
Here is a sketch of the region that we are after.

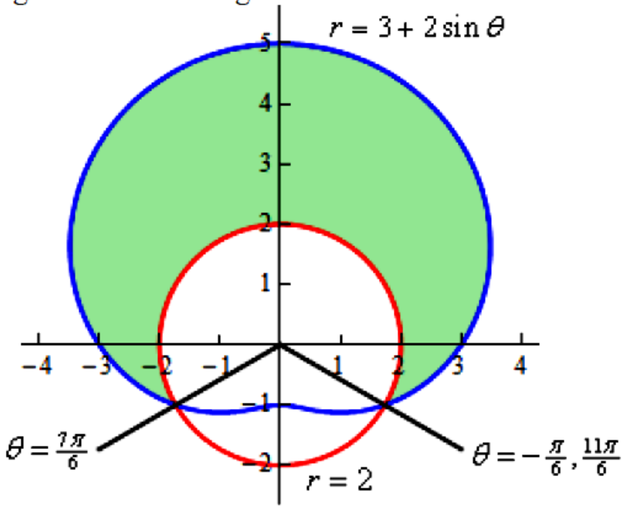


To determine this area we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3+2\sin\theta=2$$

$$\sin\theta=-\frac{1}{2} \qquad \Rightarrow \qquad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

So, the area is then,

$$A=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left((3+2\sin\theta)^2-(2)^2\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(5+12\sin\theta+4\sin^2\theta\right)d\theta$$

$$=\int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}\frac{1}{2}\left(7+12\sin\theta-2\cos(2\theta)\right)d\theta$$

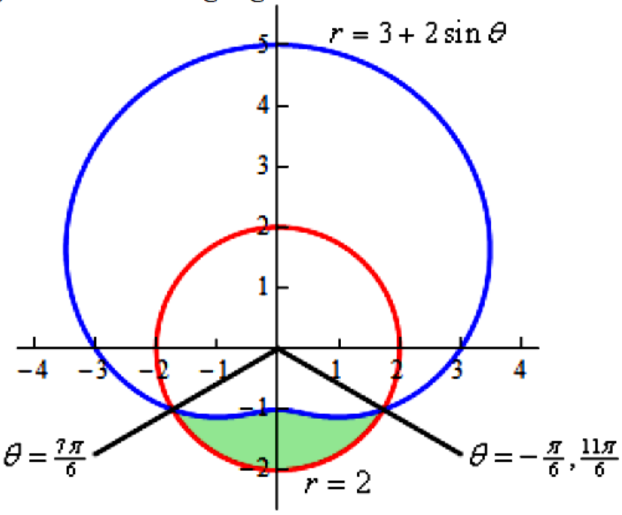
$$=\frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}}$$

$$=\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}=24.187$$

Example 3 Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

Solution

This time we're looking for the following region.



So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3 + 2\sin\theta)^2\right) d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5 - 12\sin\theta - 4\sin^2\theta\right) d\theta$$

$$= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7 - 12\sin\theta + 2\cos(2\theta)\right) d\theta$$

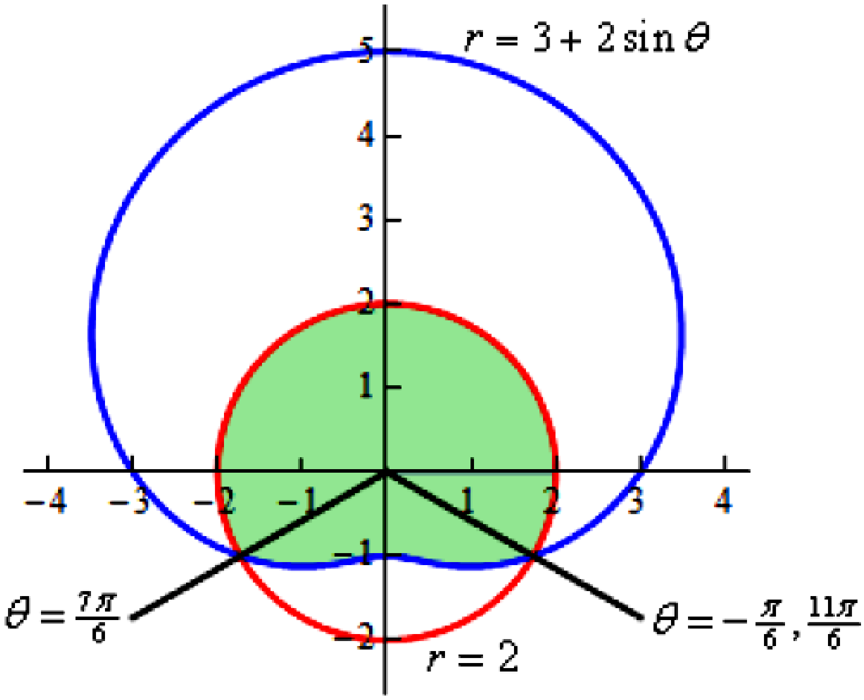
$$= \frac{1}{2}\left(-7\theta + 12\cos\theta + \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196$$

Example 4 Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution

Here is the sketch for this example.



In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1
In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned}\text{Area} &= \text{Area of Circle} - \text{Area from Example 3} \\ &= \pi(2)^2 - 2.196 \\ &= 10.370\end{aligned}$$

$$\text{Area} = \text{Area of Limacon} - \text{Area from Example 2}$$
$$= \int_0^{2\pi} \frac{1}{2}(3+2\sin\theta)^2\, d\theta - 24.187$$
$$= \int_0^{2\pi} \frac{1}{2}(9+12\sin\theta+4\sin^2\theta)\,d\theta - 24.187$$
$$= \int_0^{2\pi} \frac{1}{2}(11+12\sin\theta-2\cos(2\theta))\,d\theta - 24.187$$
$$= \frac{1}{2}(11\theta-12\cos(\theta)-\sin(2\theta))\Big|_0^{2\pi} - 24.187$$
$$= 11\pi - 24.187$$
$$= 10.370$$

Problems ***Sheet No.4***

Problems.

A- ***Polar Coordinates***.

1. For the point with polar coordinates $\left(2, \frac{\pi}{7}\right)$ determine three different sets of coordinates for the same point all of which have angles different from $\frac{\pi}{5}$ and are in the range $-2\pi \le \theta \le 2\pi$.

2. The polar coordinates of a point are $(-5, 0.23)$. Determine the Cartesian coordinates for the point.

3. The Cartesian coordinate of a point are $(2, -6)$. Determine a set of polar coordinates for the point.

4. The Cartesian coordinate of a point are $(-8, 1)$. Determine a set of polar coordinates for the point.

For problems 5 and 6 convert the given equation into an equation in terms of polar coordinates.

5. $\dfrac{4x}{3x^2+3y^2} = 6 - xy$

6. $x^2 = \dfrac{4x}{y} - 3y^2 + 2$

For problems 7 and 8 convert the given equation into an equation in terms of Cartesian coordinates.

7. $6r^3 \sin\theta = 4 - \cos\theta$

8. $\dfrac{2}{r} = \sin\theta - \sec\theta$

For problems 9 – 16 sketch the graph of the given polar equation.

9. $\cos\theta = \dfrac{6}{r}$

10. $\theta = -\dfrac{\pi}{3}$

11. $r = -14\cos\theta$

12. $r = 7$

13. $r = 9\sin\theta$

14. $r = 8 + 8\cos\theta$

15. $r = 5 - 2\sin\theta$

16. $r = 4 - 9\sin\theta$

Problems ***Sheet No.4***

B- ***Tangents with Polar Coordinates.***

1. Find the tangent line to $r = \sin(4\theta)\cos(\theta)$ at $\theta = \dfrac{\pi}{6}$.

2. Find the tangent line to $r = \theta - \cos(\theta)$ at $\theta = \dfrac{3\pi}{4}$.

C- ***Area with Polar Coordinates***.

1. Find the area inside the inner loop of $r = 3 - 8\cos\theta$.

2. Find the area inside the graph of $r = 7 + 3\cos\theta$ and to the left of the y-axis.

3. Find the area that is inside $r = 3 + 3\sin\theta$ and outside $r = 2$.

4. Find the area that is inside $r = 2$ and outside $r = 3 + 3\sin\theta$.

5. Find the area that is inside $r = 4 - 2\cos\theta$ and outside $r = 6 + 2\cos\theta$.

6. Find the area that is inside both $r = 1 - \sin\theta$ and $r = 2 + \sin\theta$.

D- ***Arc Length with Polar Coordinates.***

1. Determine the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = -4\sin\theta,\ 0 \le \theta \le \pi$$

For problems 2 and 3 set up, but do not evaluate, an integral that gives the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of θ.

2. $r = \theta\cos\theta,\ 0 \le \theta \le \pi$

3. $r = \cos(2\theta) + \sin(3\theta),\ 0 \le \theta \le 2\pi$