University of Anbar
College of Engineering
Civil Engineering Department

Fluid Mechanics

Chapter Two/ Pressure Distribution in a Fluid

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Chapter Two
pressure Distribution in a Fluid

2.1 Introduction

In static fluids, no relative motion between the fluids particles exists, therefore no velocity gradients in the fluid exist, and hence no "shear stresses" exist. Only "normal stresses (pressure)" exist. In this chapter, the pressure distribution in a static fluid and its effects on surfaces and bodies submerged or floating in it will be investigated.

Pressure is defined as a normal force exerted by a fluid per unit area. Since pressure is defined as force per unit area, it has the unit of newtons per square meter (N/m^2), which is called a pascal (Pa) [N/m^2 = Pa, Ibf/ft^2 = Psf, Ibf/in^2 = Psi].

1 bar = 105 Pa = 0.1 MPa = 100 kPa

1 atm = 101,325 Pa = 101.325 kPa = 1.01325 bars

2.2 Absolute, gage, and vacuum pressures

The actual pressure at a given position is called the absolute pressure, and it is measured relative to absolute vacuum (i.e., absolute zero pressure). Most pressure-measuring devices, however, are calibrated to read zero in the atmosphere (Figure 1), and so they indicate the difference between the absolute pressure and the local atmospheric pressure. This difference is called the gage pressure. Pressures below atmospheric pressure are called vacuum pressures and are measured by vacuum gages that indicate the difference between the atmospheric pressure and the absolute pressure. Absolute, gage, and vacuum pressures are all positive quantities and are related to each other by

$$P_{gage} = P_{abs.} - P_{atm.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.1$$

$$P_{vac.} = P_{atm.} - P_{abs.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2.2$$

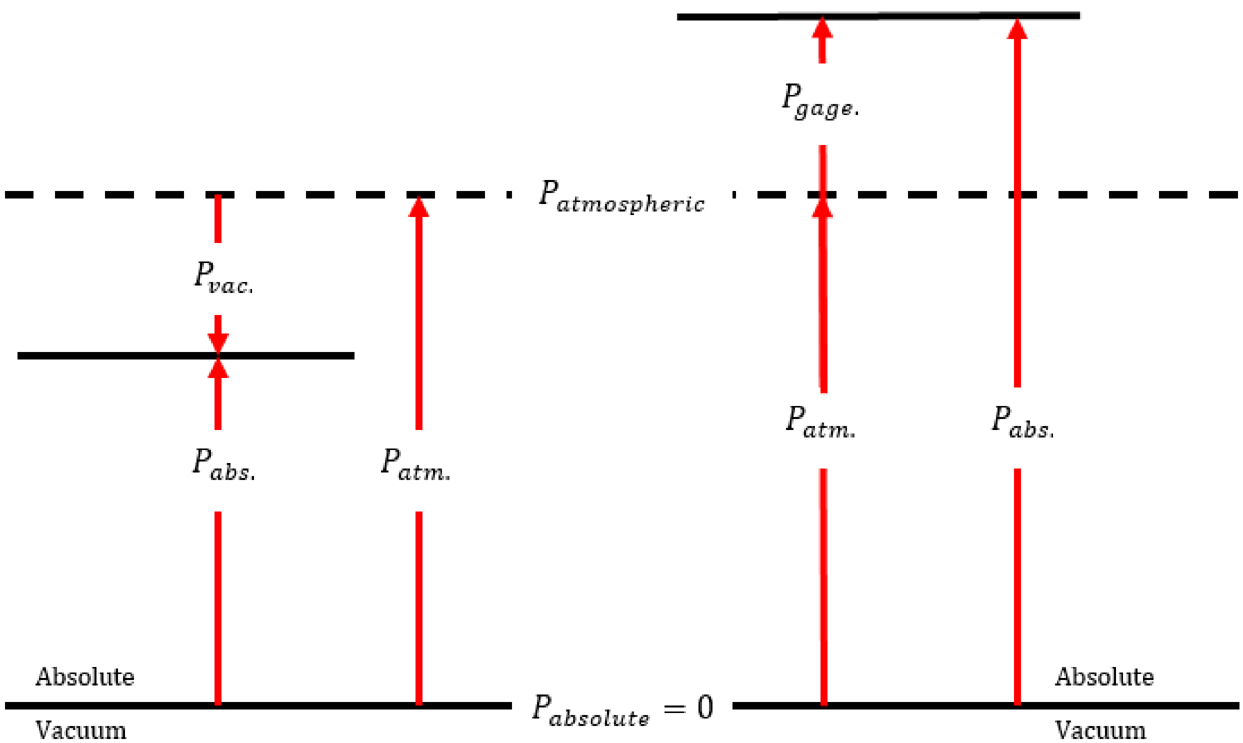


Figure 1.Relation among absolute, gage, and vacuum pressures.

Example 2.1

A vacuum gage connected to a chamber reads 5.8 psi at a location where the atmospheric pressure is 14.5 psi. Determine the absolute pressure in the chamber.

Solution: $\boldsymbol{P_{abs.} = P_{atm.} - P_{vac.} = 14.6 - 5.8 = 8.7\ psi}$

2.3 Pressure at a Point

Pressure is the compressive force per unit area, and it gives the impression of being a vector. However, pressure at any point in a fluid is the same in all directions. That is, it has magnitude but not a specific direction, and thus it is a scalar quantity. This can be demonstrated by considering a small wedge-shaped fluid element of unit length (into the page) in equilibrium, as shown in Figure 2. The mean pressures at the three surfaces are Px, Py, and Pn, and the force acting on a surface is the product of mean pressure and the surface area. From Newton's second law, a force balance in the x- and y-directions gives

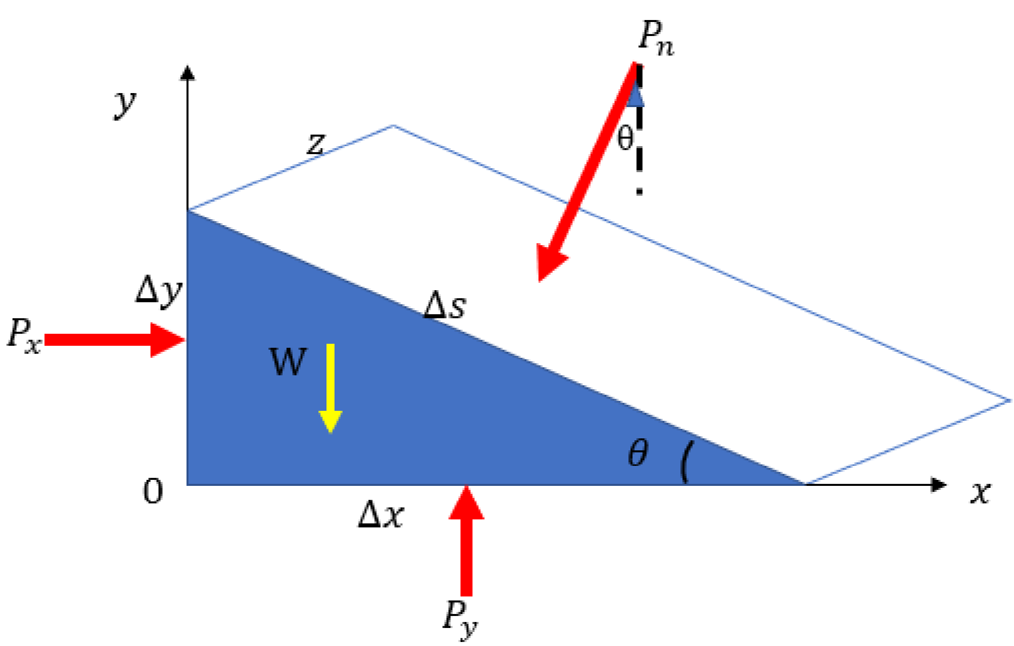


Figure 2. Equilibrium of a small wedge of fluid at rest.

$$\sum F_x = 0: \quad P_x \times \Delta y \times z - P_n \times z \times \Delta s \times \sin\theta = 0$$

$$from\ figure \qquad \Delta y = \Delta s \times \sin\theta \Longrightarrow\Longrightarrow P_x = P_n \ldots\ldots\ldots\ldots..2.3$$

$$\sum F_y = 0: \quad P_y \times \Delta x \times z - P_n \times z \times \Delta s \times \cos\theta - W_{element} = 0$$

$Also\ from\ figure \qquad \Delta x = \Delta s \times \cos\theta$

$$W_{element} = mass\ \times gravity = m \times g = \rho \times vol. \times g = \rho \times (\frac{\Delta x \Delta yz}{2}) \times g$$

where ρ is the density f the fluid element.

$$P_y \times \Delta x \times z - P_n \times z \times \Delta s \times \cos\theta - \rho \times (\frac{\Delta x \Delta yz}{2}) \times g = 0$$

The last term in Eq. $\rho \times (\frac{\Delta x \Delta yz}{2}) \times g$ drops out as Δy = 0 and the wedge becomes infinitesimal.

$P_y = P_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..2.4$

From the equation 2.3 with 2.4 we get $\boldsymbol{P_x = P_y = P_n}$

regardless of the angle θ. We can repeat the analysis for an element in the x z-plane and obtain a similar result. Thus, we conclude that **the pressure at a point in a fluid has the same magnitude in all directions**. It can be shown in the absence of shear forces that this result is applicable to fluids in motion as well as fluids at rest.

2.4 Variation of Pressure with Depth

Pressure in a fluid increases with depth because more fluid rests on deeper layers, and the effect of this "**extra weight**" on a deeper layer is balanced by an increase in pressure (Figure 3).

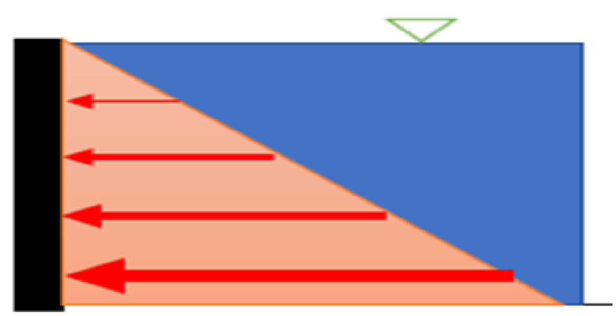
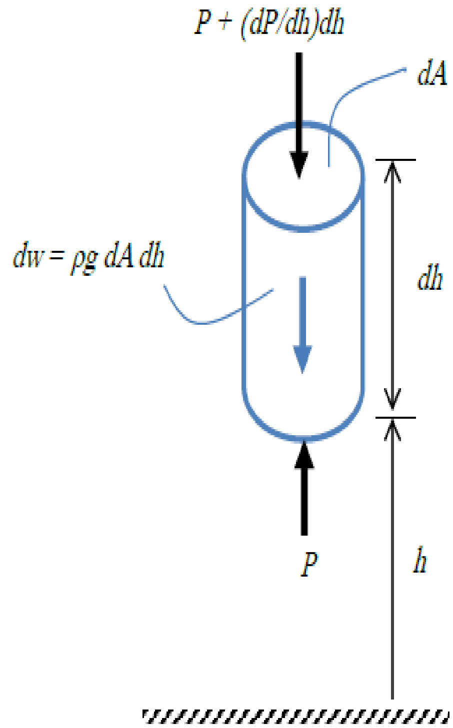


Figure 3. The pressure of a fluid at rest increases with depth

Assuming a small element with a cross sectional area ***dA*** and length ***dh*** the upward acting pressure is ***P*** and the downward acting pressure is

$P + \frac{dp}{dh} dh.$



The force balance gives:

$PdA - [P + (dP/ dh)\, dh]\, dA - \rho g dA dh = 0$

$(dP/ dh)\, dh dA = -\rho g dA dh$

$\therefore dP = -\rho g \, dh$

$$P = -\rho g \int dh = -\rho g h = -\gamma h$$

the approximate numerical value of g = 32.2 ft/sec^2 = 9.81m/sec^2.

Also to obtain a relation for the variation of pressure with depth, consider a rectangular fluid element of height Δy, length Δx, and unit depth (into the page) in equilibrium, as shown in Figure 4. Assuming the density of the fluid ρ to be constant, a force balance in the vertical z-direction gives

$$\sum F_y = ma_y = 0 : P_2\Delta x - P_1\Delta x - \rho g \Delta x \Delta y = 0$$

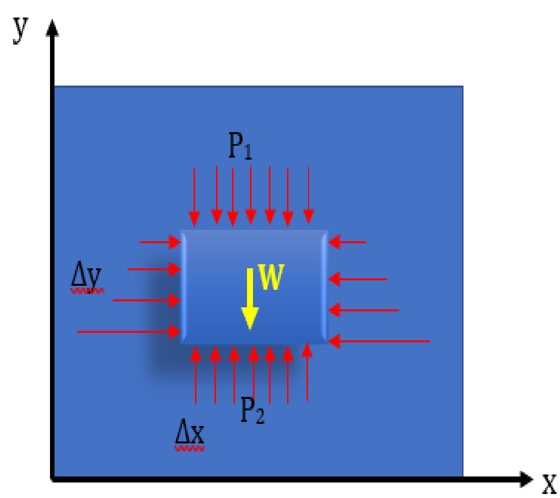


Figure 4. Free-body diagram of a rectangular fluid element in equilibrium

where $W = mg = \rho g$ Δx Δy is the weight of the fluid element. Dividing by Δx and rearranging gives $\Delta P = P_2 - P_1 = \rho g \Delta y = \gamma \Delta y$

$$\boldsymbol{P_1 = P_2 - \gamma \Delta y} \qquad or \qquad \boldsymbol{P_2 = P_1 + \gamma \Delta y}$$

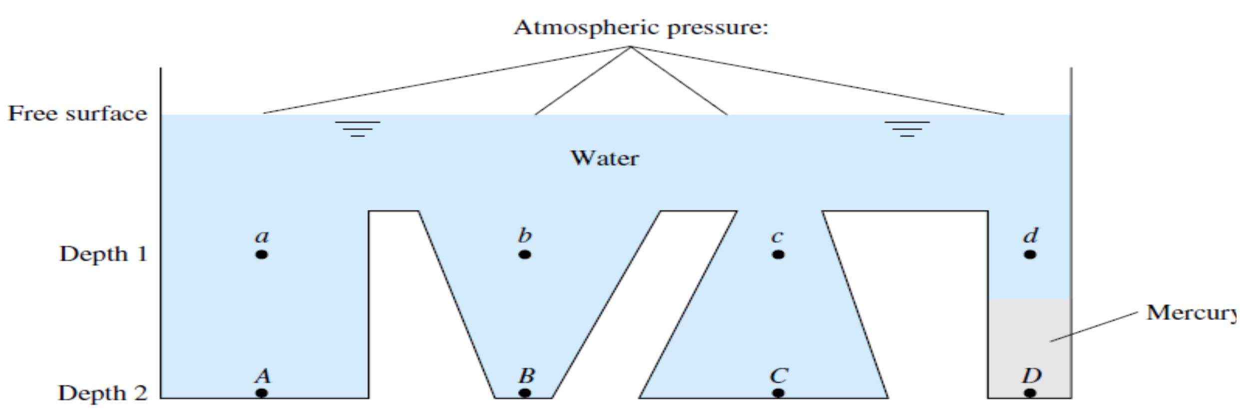
where $\gamma = \rho g$ is the specific weight of the fluid. Thus, we conclude that the pressure difference between two points in a constant density fluid is proportional to the vertical distance Δy between the points and the density ρ of the fluid.

This leads to the statement,

1. The pressure will be the same at the same level in any connected static fluid and at all points on a given horizontal plane whose density is constant or a function of pressure only.
2. The pressure increases with depth of fluid.
3. The pressure is independent of the shape of the container and the free surface of a liquid will seek a common level in any container, where the free surface is everywhere exposed to the same pressure.

We state the following conclusions about a hydrostatic condition:

Pressure in a continuously distributed uniform static fluid varies only with vertical distance and is independent of the shape of the container. The pressure is the same at all points on a given horizontal plane in the fluid. The pressure increases with depth in the fluid as shown in Figure.



Hydrostatic-pressure distribution. Points **a**, **b**, **c**, and **d** are at equal depths in water and therefore have identical pressures. Points **A**, **B**, and **C** are also at equal depths in water and have identical pressures higher than **a**, **b**, **c**, and **d**. Point **D** has a different pressure from **A**, **B**, and **C** because it is not connected to them by a water path.

2.5 Pressure Measurements

Pressure gauges can be divided into two main categories:

2.5.1 Mechanical gauges:

These are the devices in which the pressure is measured by balancing the fluid column by spring (elastic element) or dead weight. Generally, these gauges are used for measuring high pressure and where high precision is not required. Some commonly used mechanical gauges are:

2.5.1.1 Bourdon tube pressure gauge:

Bourdon tube pressure gauge is used for measuring high as well as low pressures. A simple form of this gauge is shown in Figure 5. In this case, the pressure element consists of a metal tube of approximately elliptical cross-section. This tube is bent in the form of a segment of a circle and responds to pressure changes.

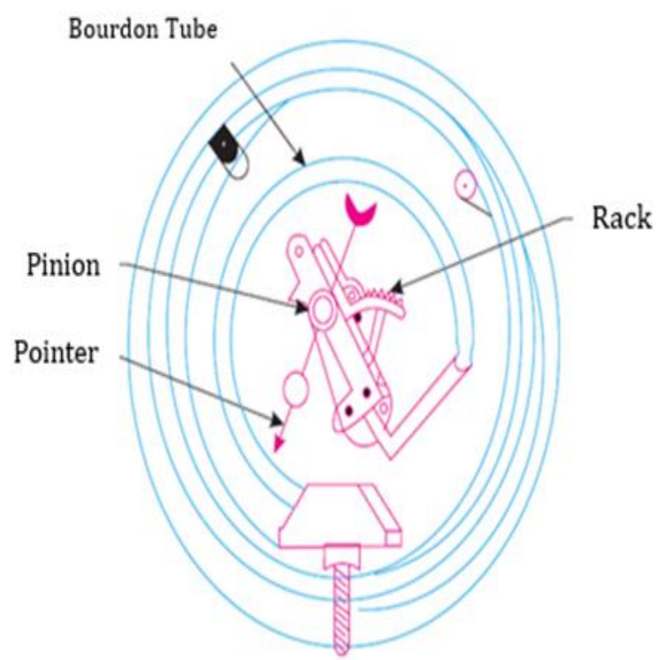


Figure 5. Bourdon tube pressure gauge.

2.5.1.2 Diaphragm gauge:

This type of gauge employs a metallic disc or diaphragm instead of a bent tube. This disc or diaphragm is used for actuating the indicating device. Shown in the Figure 6. When pressure is applied on the lower side of the diaphragm it is deflected upward. This movement of the diaphragm is transmitted to a rack and pinion. The latter is attached to the spindle of needle moving on a graduated dial. The dial can again be graduated in a suitable scale.

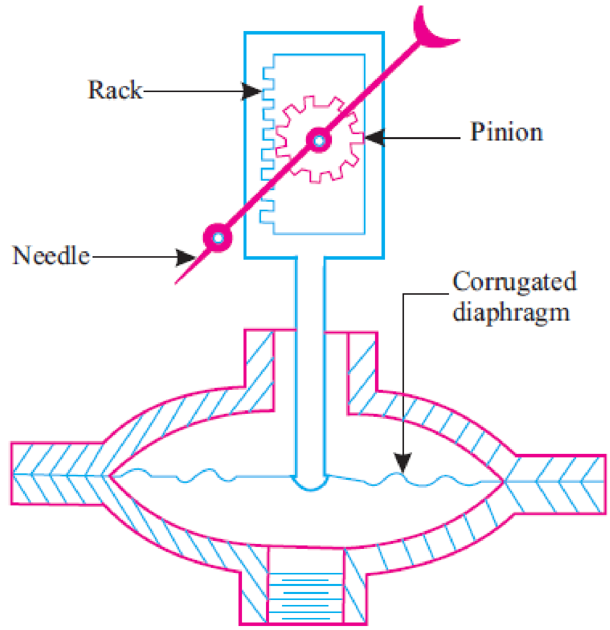


Figure 6. Diaphragm gauge.

2.5.1.3 Vacuum gauge:

Bourdon gauges discussed earlier can be used to measure vacuum instead of pressure. Slight changes in the design are required in this purpose. Thus, in this case, the tube be bent inward instead of outward as in pressure gauges. Vacuum gauges are graduated in millimeters of mercury below atmospheric pressure.

2.5.2 The Manometer

Manometer is commonly used to measure **small and moderate pressure differences**. A manometer mainly consists of a glass or plastic U-tube containing one or more fluids such as mercury, water, alcohol, or oil. To keep the size of the manometer to a manageable level, heavy fluids such as mercury are used if large pressure differences are anticipated.

2.5.2.1 Piezometer Tube

The simplest type of manometer consists of a vertical tube, open at the top, and attached to the container in which the pressure is desired, as illustrated in figure below. Since manometers involve columns of fluids at rest, the fundamental equation describing their use is Eq.

$P = P_o + \gamma h$

This gives the pressure at any elevation within a homogeneous fluid in terms of a reference pressure $\boldsymbol{P_o}$ and the vertical distance $\boldsymbol{h}$ between $\boldsymbol{p}$ and $\boldsymbol{P_o}$. Remember that in a fluid at rest pressure will increase as we move downward and will decrease as we move upward. Application of this equation to the piezometer tube of figure shown indicates that the pressure P_A can be determined by a measurement of h through the relationship

$P_A = \gamma_1 h_1$

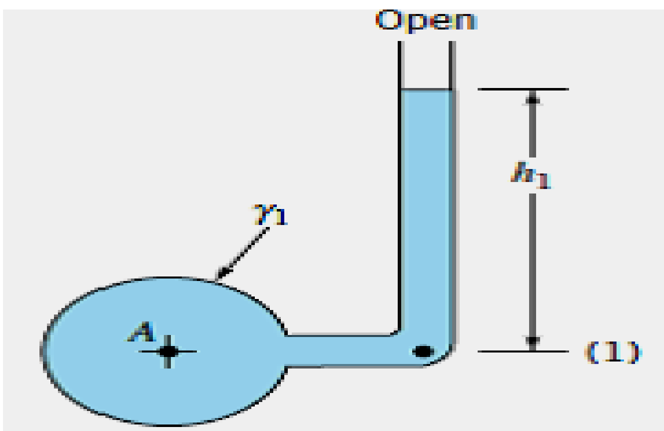


Figure 7. Piezometer Tube

The tube is open at the top, the pressure P_o can be set equal to zero as using a gauge pressure, with the height $h_1$measured from the meniscus at the upper surface to point (1) then

$$h_1 = \frac{P_A}{\rho g}$$

Consider the manometer shown in Figure 8 that is used to measure the pressure in the tank. Since the gravitational effects of gases are negligible, the pressure anywhere in the tank and at position 1 has the same value. Furthermore, since pressure in a fluid does not vary in the horizontal direction within a fluid, the pressure at point 2 is the same as the pressure at point 1,

$\mathbf{P_2 = P_1}$.

The differential fluid column of height **h** is in static equilibrium, and it is open to the atmosphere. Then the pressure at point 2 is determined directly from previous Equations to be $P_2 = P_{atm.} + \rho g h$ where **ρ** is the density of the fluid in the tube. Note that the cross-sectional area of the tube has no effect on the differential height **h**, and thus the pressure exerted by the fluid.

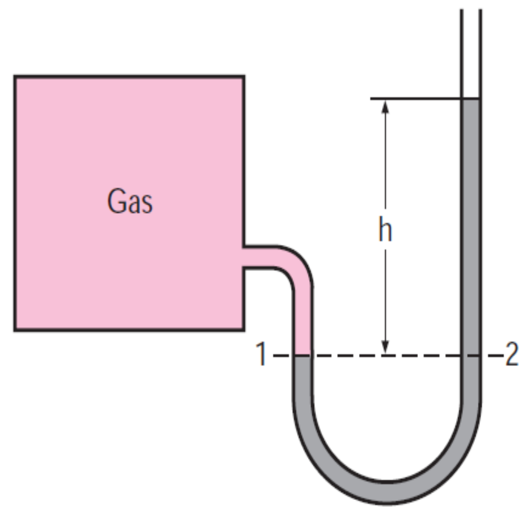
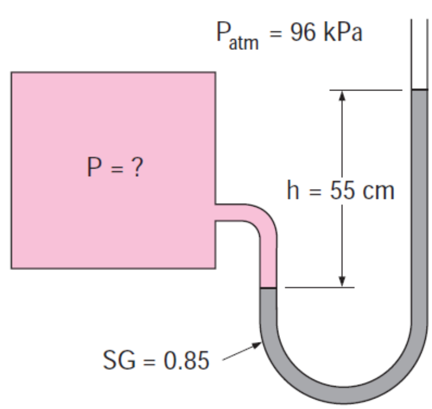


Figure 8. The basic manometer.

Example:

A manometer is used to measure the pressure in a tank. The fluid used has a specific gravity of **0.85**, and the manometer column height is **55 cm**, as shown in Figure. If the local atmospheric pressure is **96** kPa, determine the absolute pressure within the tank.



Solution:

The density of the fluid is obtained by multiplying its specific gravity by the density of water, which is taken to be 1000 kg/m^3:

ρ = SG ($\rho H2O$) = (0.85) (1000 kg/m^3) = 850 kg/m^3

$P_2 = P_{atm.} + \rho g h = 96 + (850 \times 9.81 \times 0.55) = 100.6\ kpa$

Many engineering problems and some manometers involve multiple immiscible fluids of different densities stacked on top of each other. Such systems can be analyzed easily by remembering that

(1) The pressure change across a fluid column of height h is $\Delta P = \rho g h$.

(2) Pressure increases downward in a given fluid and decreases upward (i.e., $P_{bottom} > P_{top}$).

(3) Two points at the same elevation in a continuous fluid at rest are at the same pressure.

For example, the pressure at the bottom of the tank in Figure 9 can be determined by starting at the free surface where the pressure is $\boldsymbol{P_{atm}}$, moving downward until we reach **point 1** at the bottom, and setting the result equal to $P1$. It gives

$$\boldsymbol{P_{atm.} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1}$$

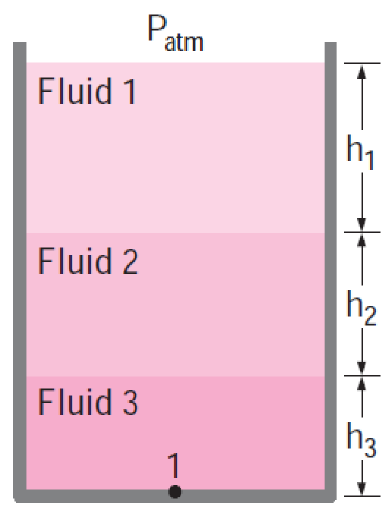
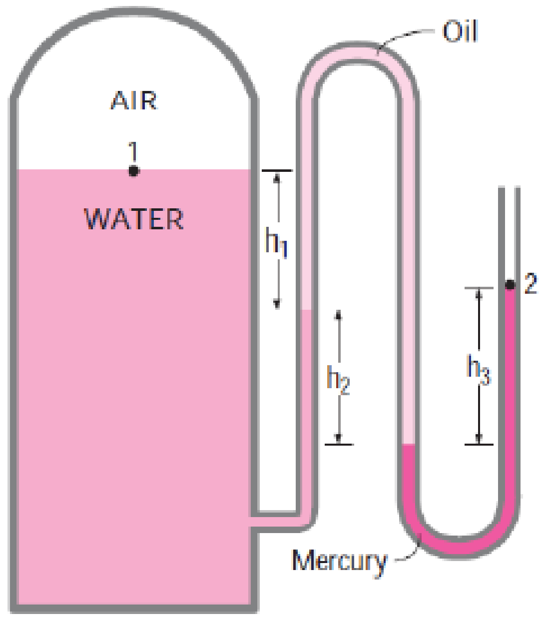


Figure 9. pressure change across a fluid layer of density ρ and height h

Example

The water in a tank is pressurized by air, and the pressure is measured by a multifluid manometer as shown in Figure. The tank is located on a mountain at an altitude of **1400 m** where the atmospheric pressure is **85.6 kPa**. Determine the air pressure in the tank if h_1 = 0.1 m, h_2 = 0.2 m, and h_3 = 0.35 m. Take the densities of water, oil, and mercury to be 1000 kg/m³, 850 kg/m³, and 13,600 kg/m³, respectively.



Solution:

Given

$P_{atm.} = 85600\ kPa$

$h_1 = 0.1\ m,\ \ h_2 = 0.2\ m,\ \ h_3 = 0.35\ m$

$\rho_w = 1000 \dfrac{kg}{m^3},\qquad \rho_{oil} = 850 \dfrac{kg}{m^3}$

$\rho_{Hg} = 13600\ kg/m^3$

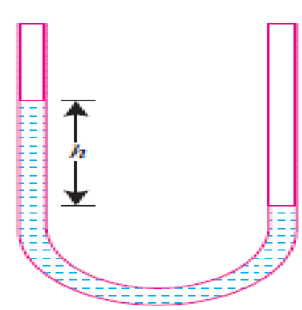
$$P_{atm} + \rho_{H.g} g h_3 - \rho_{oil} g h_2 - \rho_w g h_1 = P_1$$

$$P_1 = 85600 + (13600 \times 9.81 \times 0.35) -$$

$$(850 \times 9.81 \times 0.2) - (1000 \times 9.81 \times 0.1) =$$

Example

A manometer is used to measure the pressure drop for flow through the pipe. The difference in level was found to be 20 cm. If the manometric fluid is CCl4, find the pressure drop in S.I units (density of CCl4 = 1.596 g/cm³). If the manometric fluid is changed to mercury (ρ = 13.6gm/cm³) what will be the difference in level?



2.5.3 Atmospheric Pressure Measurement

Atmospheric pressure is measured by a device called a **barometer**; thus, the atmospheric pressure is often referred to as the barometric pressure. **Barometer** consists of a glass or Perspex tube with one open and immersed in a bath of mercury, see Figure 10. The Italian Evangelista Torricelli (1608–1647) was the first to conclusively utilize the basic barometer to measure the atmospheric pressure by writing a force balance in the vertical direction gives

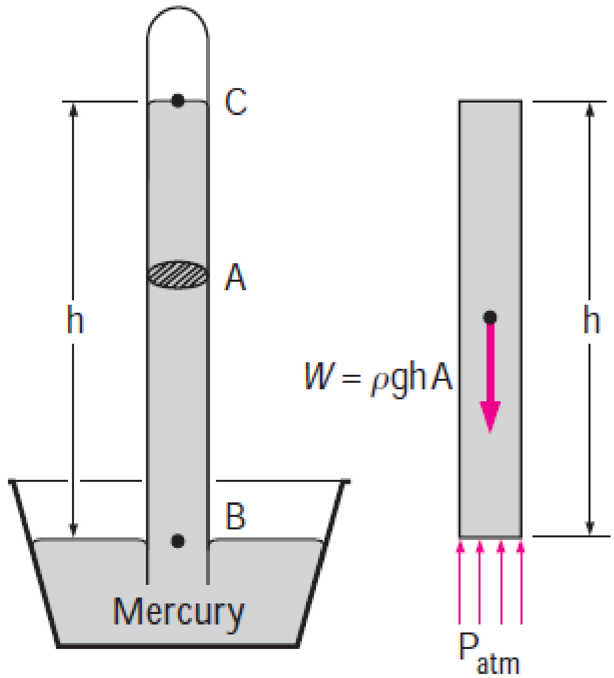


Figure 10. The basic barometer.

$F = W_{Hg} = mg = \rho_{Hg} gh\,A$ $\qquad P_{atm} = \rho gh$

A frequently used pressure unit is the standard atmosphere, which is defined as the pressure produced by a column of mercury 760 mm in height at 0°C (ρ_{Hg} = 13,595 kg/m^3) under standard gravitational acceleration (g = 9.807 m/s2). The standard atmospheric pressure, for example, is 760 mmHg at 0°C. The unit mmHg is also called the torr in honor of Torricelli. Therefore, 1 atm = 760 torr and 1 torr = 133.3 Pa.

Example

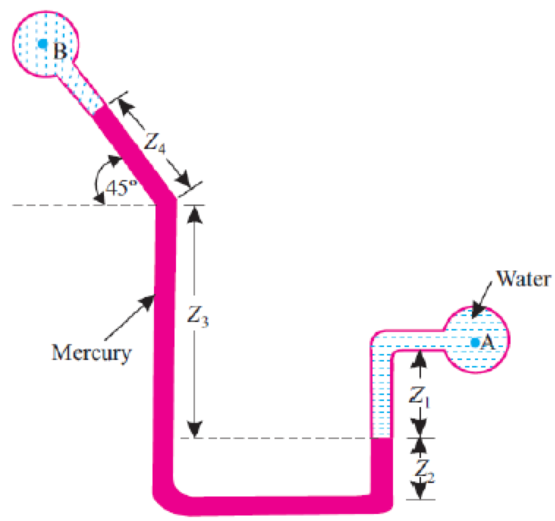
Determine the atmospheric pressure at a location where the barometric reading is 740 mm Hg and the gravitational acceleration is g = 9.81 m/s2. Assume the temperature of mercury to be 10°C, at which its density is 13,570 kg/m^3.

2.5.4 Inclined Manometer

The inclined manometer is frequently used for measuring small difference in gage pressure. It is adjusted to read zero, by moving the inclined scale. Since the inclined tube requires a greater displacement of the meniscus for given pressure difference than a vertical tube, it offers greater accuracy in reading the scale.

Example:

For the Figure, determine the pressure difference between pipes A and B. Take Z_1= 0.45 m, Z_2= 0.225 m, Z_3= 0.675 m, Z_4= 0.3 m. Neglect pressure due to pressure of air column in the inclined tube.



2.6 Hydrostatic Forces on Submerged Plane Surfaces

The plane of this surface (normal to the page) intersects the horizontal free surface with an **angle θ**, and we take the line of intersection to be the x-axis as shown Figure 11. The absolute pressure above the liquid is **Po**, which is the local atmospheric pressure $\mathbf{P_{atm}}$ if the liquid is open to the atmosphere (but **Po** may be different than $\mathbf{P_{atm}}$ if the space above the liquid is evacuated or pressurized).

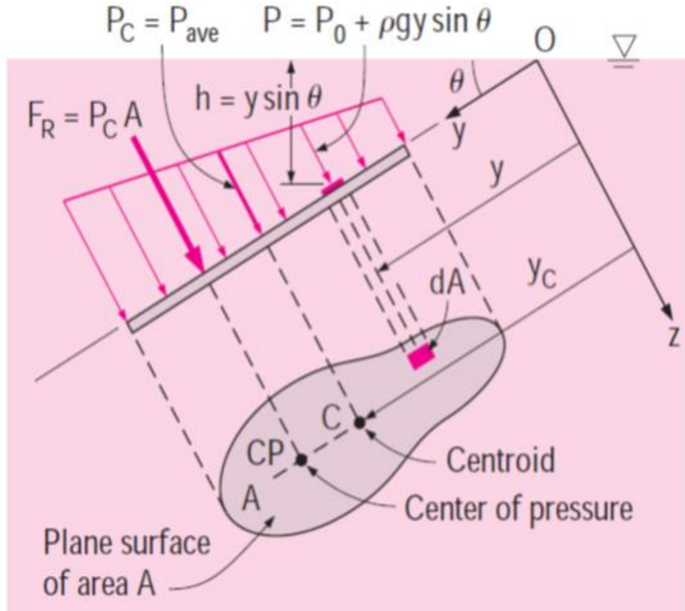


Figure 11. Hydrostatic force on an inclined plane surface completely submerged in a liquid.

Then the absolute pressure at any point on the plate is

P = Po + ρgh = Po + ρg (y sinθ)

The resultant hydrostatic force ***F*R** acting on the surface is determined by integrating the force ***P dA*** acting on a differential area ***dA*** over the entire surface area,

$$F_R = \int_A P\, dA = \int_A (P_o + \rho g \sin\theta) dA = P_o A + \rho g \sin\theta \int_A y dA$$

But the first moment of area $\left(\int_A y dA\right)$ is related to the y-coordinate of the centroid (or center) of the surface by $y_c = \frac{1}{A}\int_A y dA$ Substituting,

$$F_R = (P_o + \rho g y_c \sin\theta)\, A = (P_o + \rho g h_c) A + P_c A = P_{Ave} A$$

where $P_c = P_o + \rho g h_c$ is the pressure at the centroid of the surface, which is equivalent to the average pressure on the surface, and $h_c = y_c \sin\theta$ is the vertical distance of the centroid from the free surface of the liquid Figure 12

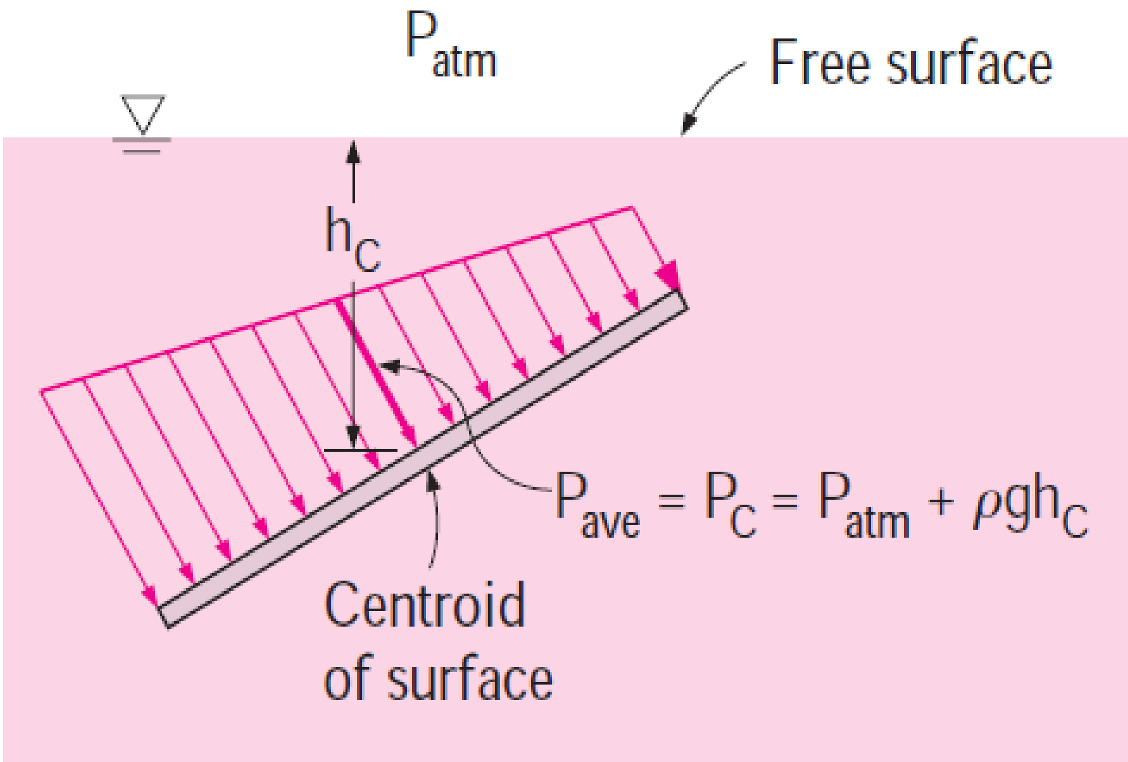


Figure 12. The pressure at the centroid of a surface is equivalent to the average pressure on the surface.

Thus, we conclude that:

The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure Pc at the centroid of the surface and the area A of the surface (see Figure 13).

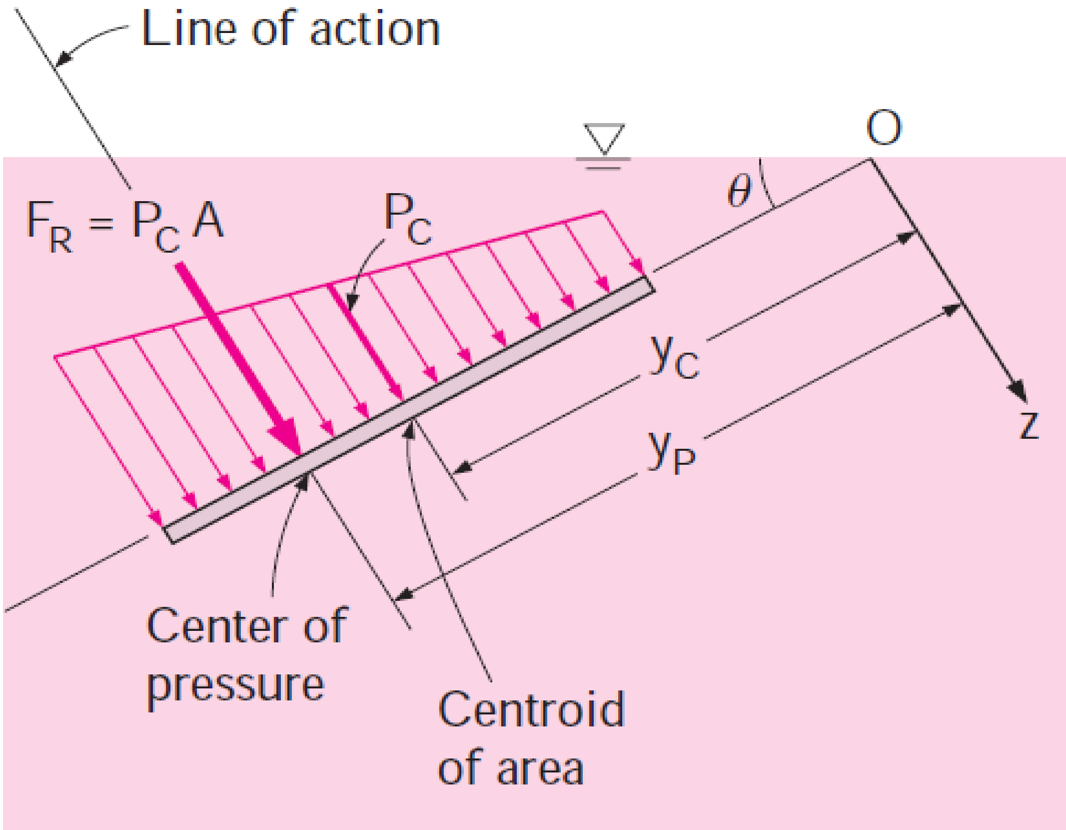


Figure 13. The resultant force acting on a plane surface is equal to the product of the pressure at the centroid of the surface and the surface area, and its line of action passes through the center of pressure.

The point of intersection of the line of action of the resultant force and the surface is the center of pressure. The vertical location of the line of action is determined by equating the moment of the resultant force to the moment of the distributed pressure force about the x-axis. It gives

$$y_p F_R = \int_A y\ P\ dA = \int_A y(P_o + \rho g \sin\theta)dA = P_o \int_A y dA + \rho g \sin\theta \int_A y^2\ dA$$

Or $\quad y_p F_R = P_o y_c\ A + \rho g \sin\theta\ I_{xx,O}$

where y_P is the distance of the center of pressure from the x-axis (point O in Figure 2.16) and is the second moment of area (also called the area moment of inertia) about the x-axis. The second moments of area are widely available for common shapes in engineering handbooks, but they are usually given about the axes passing through the centroid of the area.

The second moments of area about two parallel axes are related to each other by the parallel axis theorem, which in this case is expressed as

$$I_{xx,o} = I_{xx,c} + y^2 A$$

where $\mathbf{I_{xx,c}}$ is the second moment of area about the x-axis passing through the centroid of the area and $\mathbf{y_c}$ (the y-coordinate of the centroid) is the distance between the two parallel axes. Substituting the $\mathbf{F_R}$ relation from Equation (2.16) and the $\mathbf{I_{xx,O}}$ relation from Equation (2.19) into Equation (2.18) and solving for $\mathbf{y_P}$ gives

$$y_p = y_c + \frac{I_{xx,c}}{[y_c + P_o/(\rho g \sin\theta)]A}$$

For $P_o = 0$, which is usually the case when the atmospheric pressure is ignored, it simplifies to

$$y_p = y_c + \frac{I_{xx,c}}{y_c A}$$

Knowing $\mathbf{y_P}$, the vertical distance of the center of pressure from the free surface is determined from $h_P = y_P \sin\theta$. The $I_{xx,c}$ values for some common areas are given in Figure 2.17.

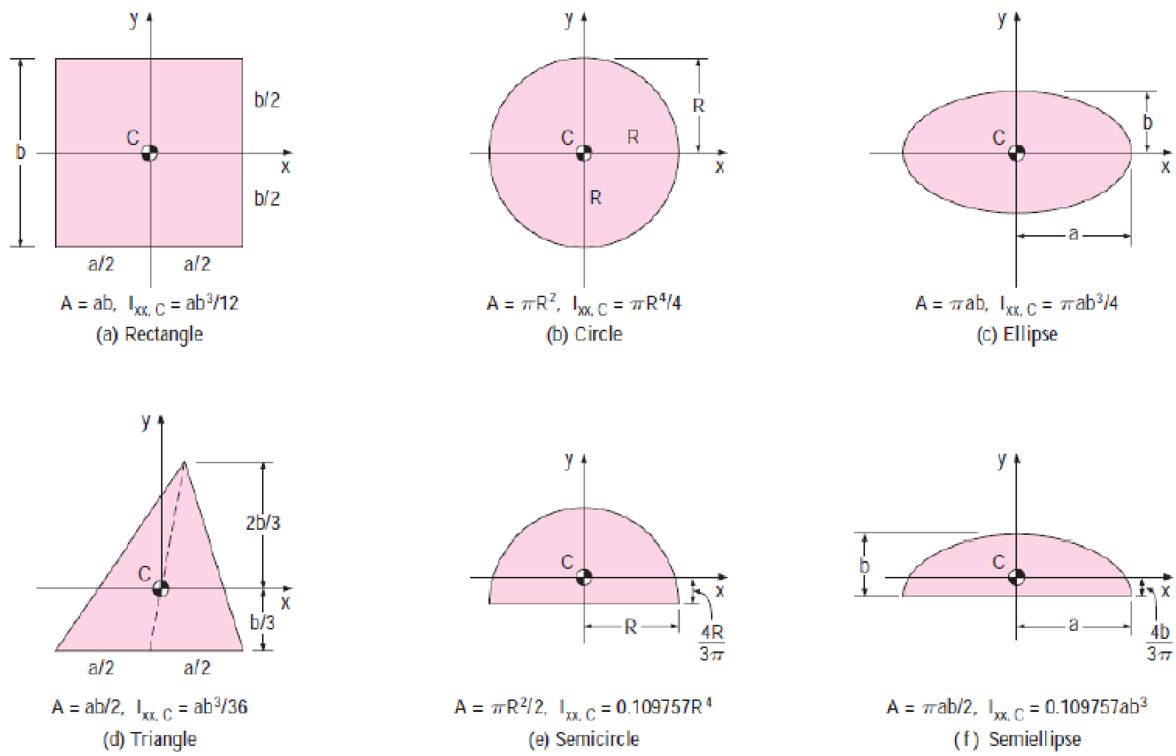


Figure 14. The centroid and the centroidal moments of inertia for some common geometries.

Example:

A circular plate 1.5 m diameter is submerged in water with its greatest and least depths below the surface being 2 m and 0.75 m respectively as shown in Figure 15. Determine: (i) The total pressure force on one face of the plate. (ii) The position of the center of pressure.

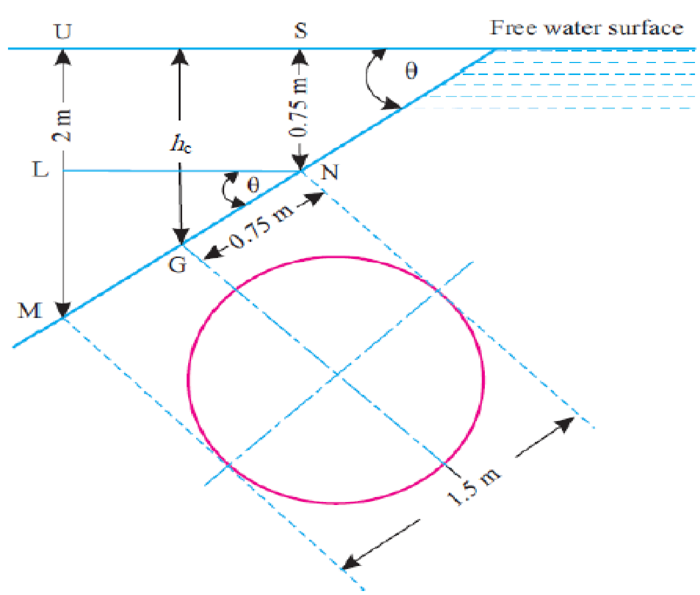


Figure 15. Schematic for Example

Solution:

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}1.5^2 = 1.767\ m^2$$

Distance of center of gravity from free surface

$$h_c = SN + GN\ \sin\theta, \quad = 0.75 + 0.75\ \sin\theta$$

$$\sin\theta = \frac{LM}{MN} = \frac{UM - UL}{MN} = \frac{2 - 0.75}{1.5} = 0.8333 \quad \Rightarrow h_c = 0.75 + 0.75 \times 0.8333 = 1.375\ m$$

i) Total pressure force (P):

$$F_p = \rho_w g A h_c = 1000 \times 9.81 \times 1.767 \times 1.375 = 23830\ N$$

ii) The center of pressure (hp)

$$y_p = \frac{I_{xx,c}}{Ay_c} + y_c, \quad h_c = y_c \sin\theta, \quad \Rightarrow y_c = \frac{h_c}{\sin\theta} \quad and \quad y_p = \frac{h_p}{\sin\theta}$$

$$\frac{h_p}{\sin\theta} = \frac{I_{xx,c}\sin\theta}{Ah_c} + \frac{h_c}{\sin\theta} \qquad (\times \sin\theta)$$

$$h_p = \frac{I_{xx,c}sin^2\theta}{Ah_c} + h_c \quad = \frac{\frac{\pi}{64} \times 1.5^4 \times 0.8333^2}{1.767 \times 1.375} + 1.375 = 1.446\ m$$

Example:

A tank of oil has a right-triangular panel near the bottom, as shown in Figure 16. Neglecting Pa, find the (a) Hydrostatic force and (b) The location of pressure center on the panel.

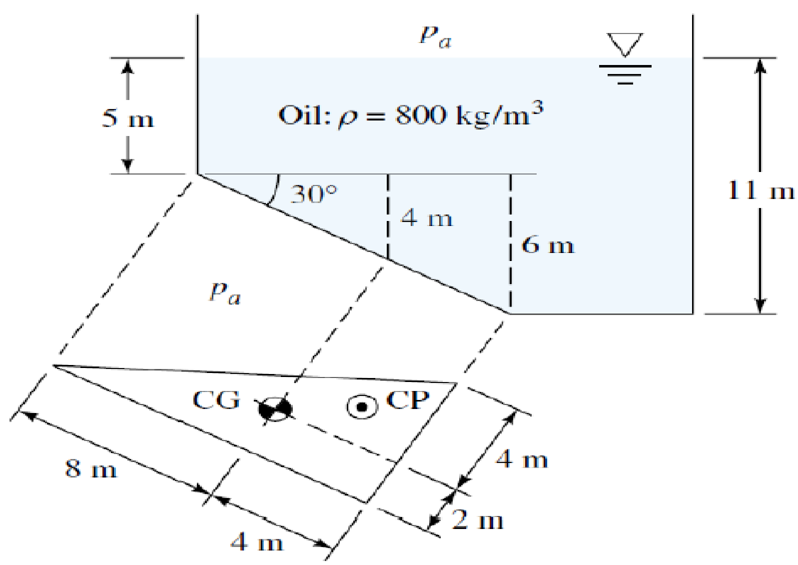


Figure 16. Schematic for Example

Solution:

The centroid is one-third up (4 m) and one-third over (2 m) from the lower left corner, as shown. The area is

$$\frac{1}{2} \times 6 \times 12 = 36\, m^2$$

The moments of inertia are

$$I_{xx} = \frac{bh^3}{36} = \frac{6 \times 12^3}{36} = 288\, m^4, \quad and \quad I_{xy} = \frac{b(b-2s)L^2}{72} = \frac{6[6-2(6)]12^2}{72} = 72\, m^4$$

The depth to the centroid is $h_{CG} = 5 + 4 = 9\, m$; thus the hydrostatic force is

$$F = \rho g h_{CG} A = 800 \times 9.81 \times 9 \times 36 = 2.54 MN$$

The position of pressure center on the panel is given as,

$$y_{cp} = -\frac{I_{xx} \sin\theta}{h_{CG} A} = -\frac{288 \times \sin 30}{9 \times 36} = -0.444\, m$$

$$x_{cp} = -\frac{I_{xy} \sin\theta}{h_{CG} A} = -\frac{(-72) \times \sin 30}{9 \times 36} = +0.111\, m$$

Example:

The gate in Figure 17is 5 ft wide, is hinged at point B, and rests against a smooth wall at point A. Compute (a) the force on the gate due to seawater pressure, (b) the horizontal force P exerted by the wall at point A, and (c) the reactions at the hinge B.

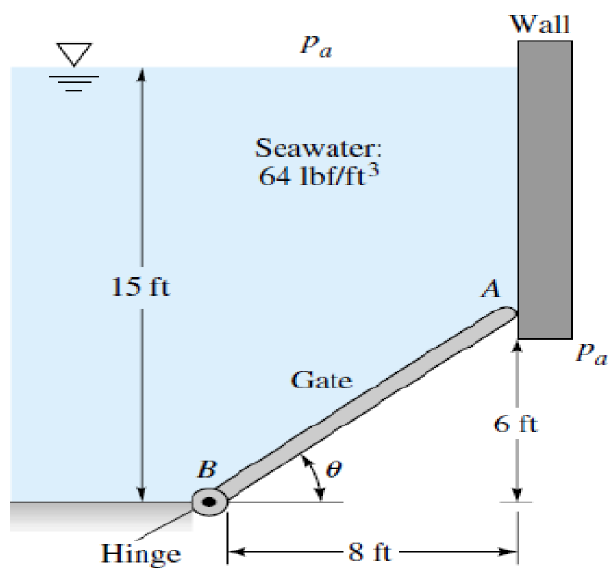


Figure 17. Schematic for Example

Solution:

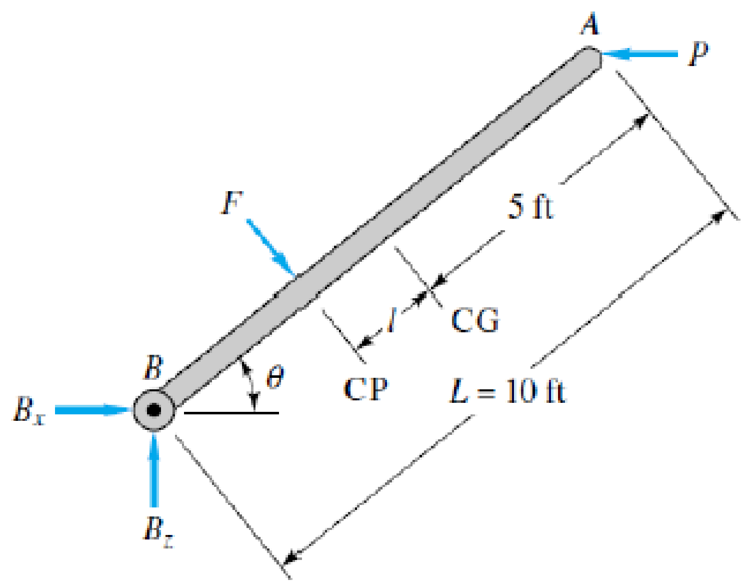
By geometry the gate is 10 ft long from A to B, and its centroid is halfway between, or at elevation 3 ft above point B. The depth h_{CG} is thus $[15 - 3] = 12\ ft$. The gate area is$[5(10) = 50 ft^2]$. Neglect pa as acting on both sides of the gate. The hydrostatic force on the gate is

$$F = P_{CG}A = \gamma h_{CG}A = 64 \times 12 \times 50 = 38400 Ibf$$

First, we must find the center of pressure of F. A free-body diagram of the gate is shown in the Figure. The gate is a rectangle, hence

$$I_{xy} = 0 \quad and \quad I_{xx} = \frac{bh^3}{12} = \frac{5 \times 10^3}{12} = 417 ft^4$$

The distance l from the CG to the CP is given as below since Pa is neglected.



$$I = -y_{cp} = +\frac{I_{xx}\sin\theta}{h_{CG}A} = \frac{417 \times \frac{6}{10}}{12 \times 50} = 0.417\ ft$$

The distance from point **B** to force **F** is thus [10 - *l* - 5 = 4.583 ft]. Summing moments counterclockwise about **B** gives

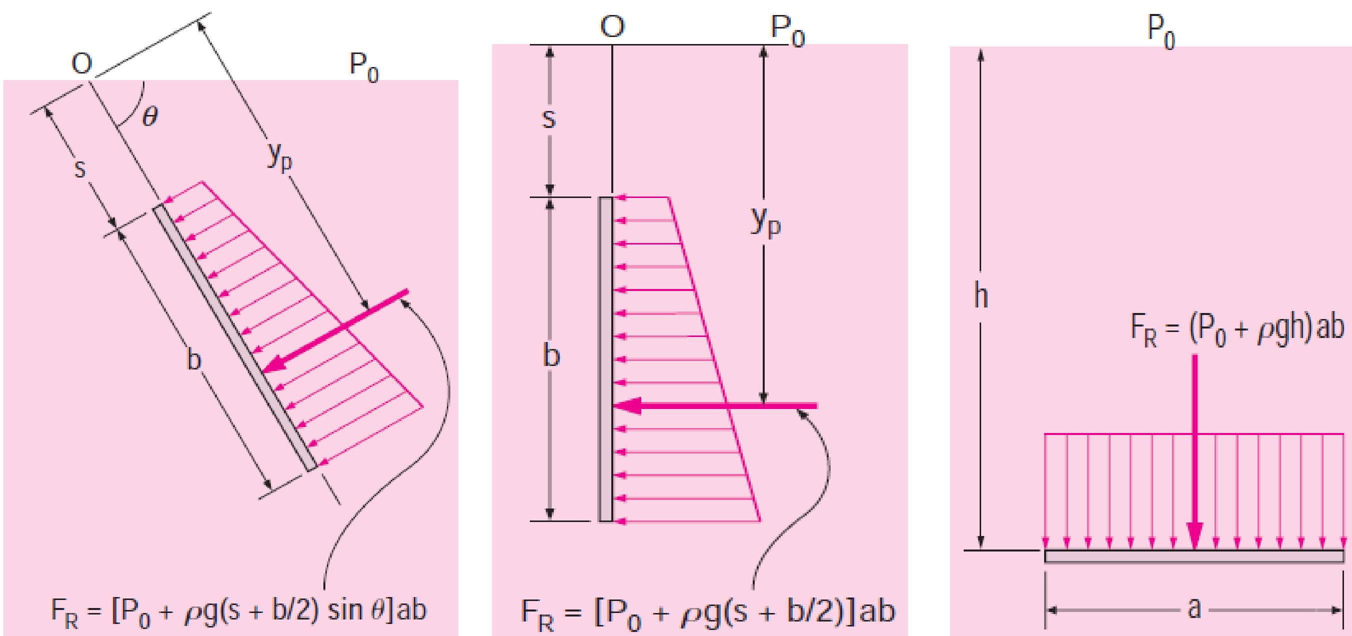
$$PL\sin\theta - F(5 - l) = P \times 6 - 38400 \times 4.583 = 0 \qquad \Longrightarrow P = 29300 Ibf$$

With **F** and **P** known, the reactions B_x and B_z are found by summing forces on the gate $\sum F_x = 0 = B_x + F\sin\theta - P = B_x + 38400 - 29300 \implies B_x = 6300\, Ibf$

$$\sum F_z = 0 = B_z + F\cos\theta - P = B_z - 38400 \times 0.8 \implies B_z = 30700 Ibf$$

2.6.1 Special Case: Submerged Rectangular Plate

Consider a completely submerged rectangular flat plate of height **b** and width **a** tilted at an **angle θ** from the horizontal and whose top edge is horizontal and is at a distance **(s)** from the free surface along the plane of the plate, as shown in Figure 18. The resultant hydrostatic force on the upper surface is equal to the average pressure, which is the pressure at the midpoint **(CG)** of the surface, times the surface area **A**. That is,



(a) (b) (c)

Figure 18: Hydrostatic force acting on the top surface of a submerged rectangular plate for tilted, vertical, and horizontal cases.

2.6.1.1 Tilted rectangular plate:

$F_R = P_C A = [P_o + \rho g(s + b/2)\sin\theta]ab$

The force acts at a vertical distance of $[h_p = y_p \sin\theta]$ from the free surface directly beneath the centroid of the plate where,

$$y_p = s + \frac{b}{2} + \left[\frac{ab^3/12}{\left[s + \frac{b}{2} + \frac{P_o}{\rho g \sin\theta}\right] ab}\right] \Longrightarrow y_p = s + \frac{b}{2} + \left[\frac{b^2}{\left[s + \frac{b}{2} + \frac{P_o}{\rho g \sin\theta}\right] 12}\right]$$

When the upper edge of the plate is at the free surface and thus s = 0, previous equation $\mathbf{y_p}$ reduces to

$(s = 0) \Longrightarrow F_R = [P_o + \rho g(b\sin\theta)/2]ab$

2.6.1.2 Vertical rectangular plate

For a completely submerged vertical plate $(\theta = 90)$ whose top edge is horizontal, the hydrostatic force can be obtained by setting $\sin\theta = 1$ see figure 18 (b). (for more details).

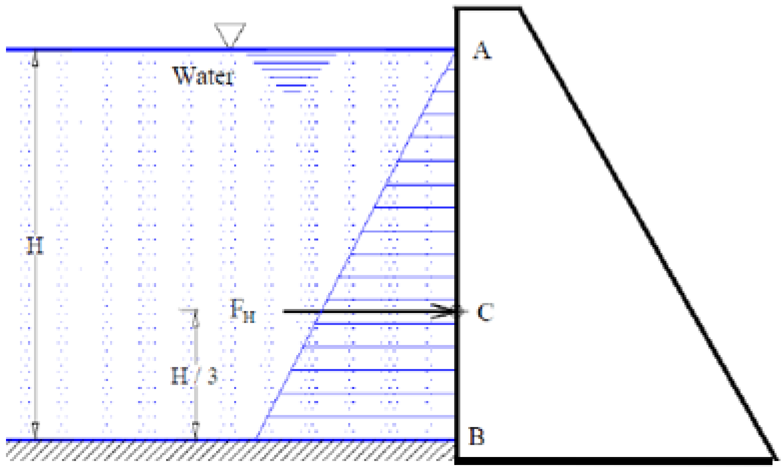
Vertical rectangular plate: $F_R = P_C A = [P_o + \rho g(s + b/2)]ab$

Vertical rectangular plate (s = 0): $F_R = [P_o + \rho g b/2]ab$

When the effect of $\mathbf{P_o}$ is ignored since it acts on both sides of the plate, the hydrostatic force on a vertical rectangular surface of height **b** whose top edge is horizontal and at the free surface is $[\boldsymbol{F_R = \rho g a b^2/2}]$ acting at a distance of **2b/3** from the free surface directly beneath the centroid of the plate.

- **Supplementary Notes**

a. The surface intersects with fluid surface



Force magnitude: $F_H = \gamma * hc * A$

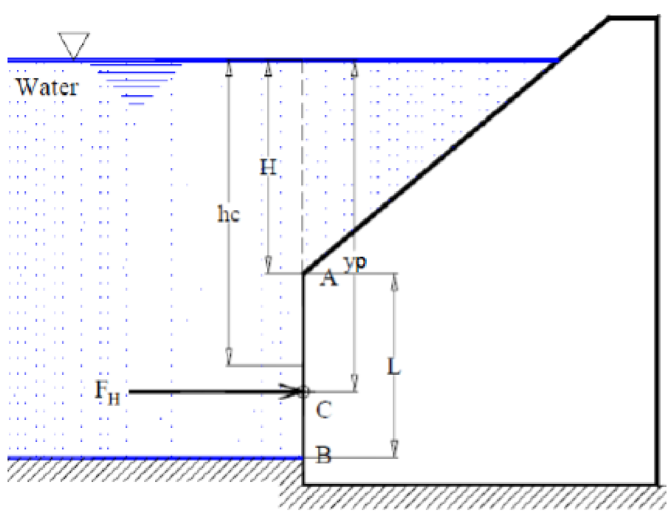
Where cross sectional area of the plan surface (= distance AB × width).

$$F_H = \frac{1}{2} \times \gamma \times H^2$$

Point of action (location):

The line of action of the force is at distance H/3 from the bottom.

b. The plan surface does not intersect with the fluid surface:



Force magnitude: $FH = \gamma * hc * A$, $h_c = H + \frac{L}{2}$

Point of action (location):

The line of action of the force is at distance = yp, from the fluid surface:

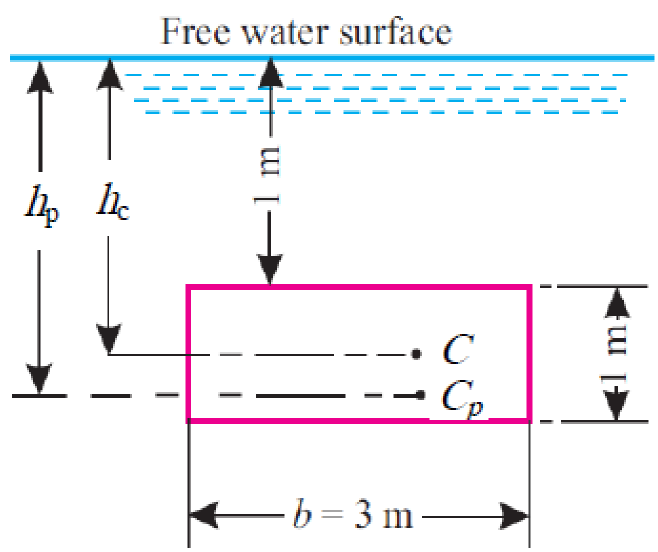
$$y_p = y_c + \frac{I_{xx}}{y_c \times A}$$

2.6.1.3 Horizontal rectangular plate:

The pressure distribution on a submerged horizontal surface is uniform, and its magnitude is ***[P = P₀ + ρgh]***, where ***h*** is the distance of the surface from the free surface. And $\boldsymbol{F_R}$ ***= [Po + ρgh] ab,*** it acts through the midpoint of the plate (see Figure 18 (c) for more details).

Example:

A rectangular plate 3 m long, 1 m deep and 1 m wide is immersed vertically in water in such a way that its 3 m side is parallel to the water surface and is 1 m below it as shown in Figure. Find (a) Total pressure on the plate (b) Position of center of pressure.



Solution:

length of the plane surface, b = 3m

Depth of the plane surface, d = 1m

Area of the plane surface, A= b × d = 3 × 1= 3m²

So, h_c= 1 + 0.5 = 1.5 m

(a) Total pressure force

P = ρ g A h_c = 9.81 × 1000 × 3 × 1.5 = 44140 N= 44.14 kN

(b) Centre of pressure, h_p

Notice: for horizontal rectangular plate, y_c= h_c, y_p= h_p and $I_{xx,o}$= $I_{xx,c}$

So $y_p = \frac{I_{xx,o}}{Ay_c} + y_c \quad will\ be\ as \quad h_p = \frac{I_{xx,c}}{Ah_c} + h_c$

$$I_{xx,c} = \frac{bd^3}{12} = \frac{3\times 1^3}{12} = 0.25\ m^4 \quad , \quad h_p = \frac{0.25}{3\times 1.5} = 1.556\ m$$

Example:

A 3-m-high, 6-m-wide rectangular gate is hinged at the top edge at A and is restrained by a fixed ridge at B as shown in Figure. Determine the hydrostatic force exerted on the gate by the 5-m-high water and the location of the pressure center.

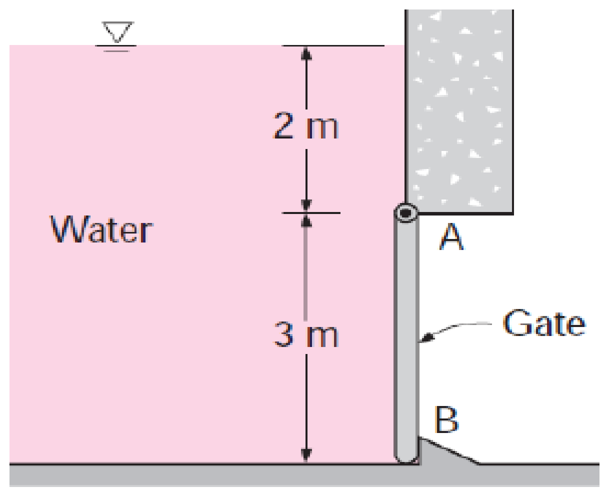


Figure 19: Schematic for Example

Solution

The average pressure on a surface is the pressure at the centroid (midpoint) of the surface, and multiplying it by the plate area gives the resultant hydrostatic force on the gate,

F_R= P A=ρ g h_c A $= 1000 \times 9.81 \times 3.5 \times 3 \times 6 = 618000\ N$

The vertical distance of the pressure center from the free surface of water is

$$y_p = s + \frac{b}{2} + \left[\frac{b^2}{\left[s + \frac{b}{2}\right] 12}\right] = 2 + \frac{3}{2} + \frac{3^2}{12(2 + 3/2)} = 3.71\ m$$

2.6.2 Hydrostatic forces on submerged curved surfaces

For a submerged curved surface, the determination of the resultant hydrostatic force is more involved since it typically requires the integration of the pressure forces that change direction along the curved surface.

The easiest way to determine the resultant hydrostatic force $\mathbf{F_R}$ acting on a two-dimensional curved surface is toP determine the horizontal and vertical components $\mathbf{F_H}$ and $\mathbf{F_V}$ separately. This is done by considering the free-body diagram of the liquid block enclosed by the curved surface and the two plane surfaces (one horizontal and one vertical) passing through the two ends of the curved surface, as shown in Figure 20.

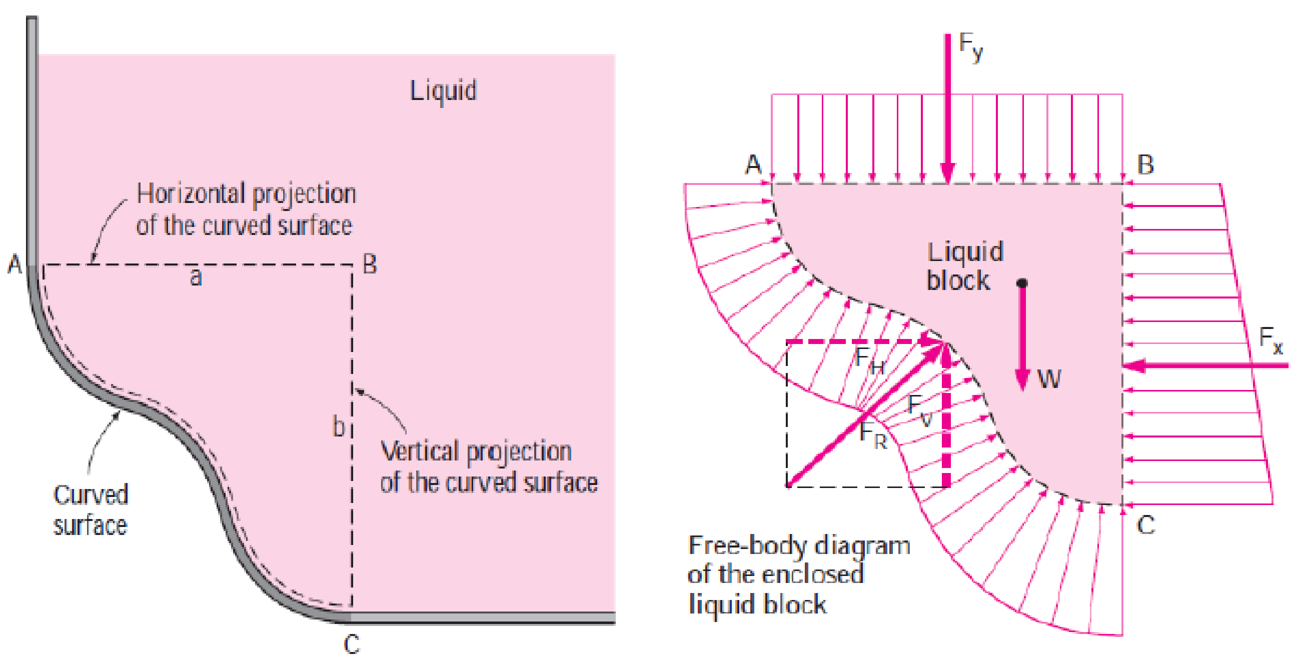


Figure 20: Determination of the hydrostatic force acting on a submerged curved surface.

The weight of the enclosed liquid block of volume **V** is simply **W = ρ g V**, and it acts downward through the centroid of this volume. Noting that the fluid block is in static equilibrium, the force balances in the horizontal and vertical directions give

- **Horizontal force component on curved surface: $F_H = F_x$**
- **Vertical force component on curved surface: $F_V = F_y + W$**

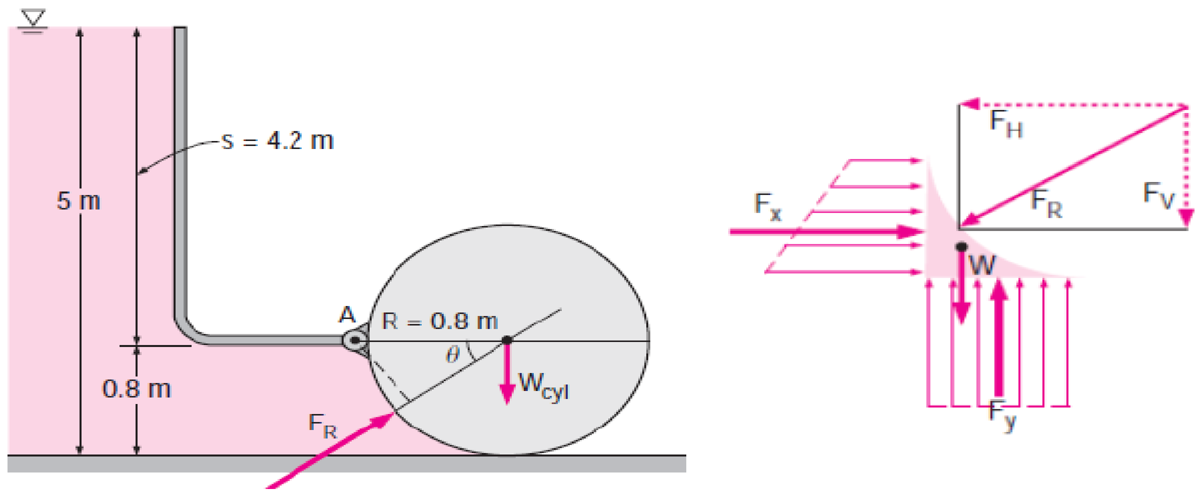
where the summation **[F_y + W]** is a vector addition (i.e., add magnitudes if both act in the same direction and subtract if they act in opposite directions). The magnitude of the resultant hydrostatic force acting on the curved surface is,

$$\boldsymbol{F_R} = \sqrt{(\boldsymbol{F_H})^2 + (\boldsymbol{F_V})^2}$$

and the tangent of the angle it makes with the horizontal is $\boldsymbol{\alpha} = \boldsymbol{F_V}/\boldsymbol{F_H}$. The exact location of the line of action of the resultant force (e.g., its distance from one of the end points of the curved surface) can be determined by taking a moment about an appropriate point.

Example:

A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate, as shown in Figure. When the water level reaches 5 m, the gate opens by turning about the hinge at point A. **Determine (a)** the hydrostatic force acting on the cylinder and its line of action when the gate opens and **(b)** the weight of the cylinder per m length of the cylinder.



Horizontal force on vertical surface:

$$F_H = F_X = P_{ave}A = \rho g h_c A = \rho g(s + R/2)A$$
$$= 10^3 \times 9.81 \times (4.2 + 0.8/2) \times 0.8 \times 1 = 36100\ N$$

Vertical force on horizontal surface (upward):

$$F_V = F_Y = P_{ave}A = \rho g h_c A = \rho g h_{bottom} A = 10^3 \times 9.81 \times 5 \times 0.8 \times 1$$
$$= 37900\ N$$

Weight of fluid block per m length (downward):

$$W = mg = \rho g v = \rho g(R^2 - \pi R^2/4) \times 1 = 10^3 \times 9.81 \times 0.8^2 \times (1 - \pi/4) \times 1$$
$$= 1300\ N$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become,

$$F_R = \sqrt{(F_H)^2 + (F_V)^2} = \sqrt{(36.1)^2 + (37.9)^2} = 52.3\ kN$$

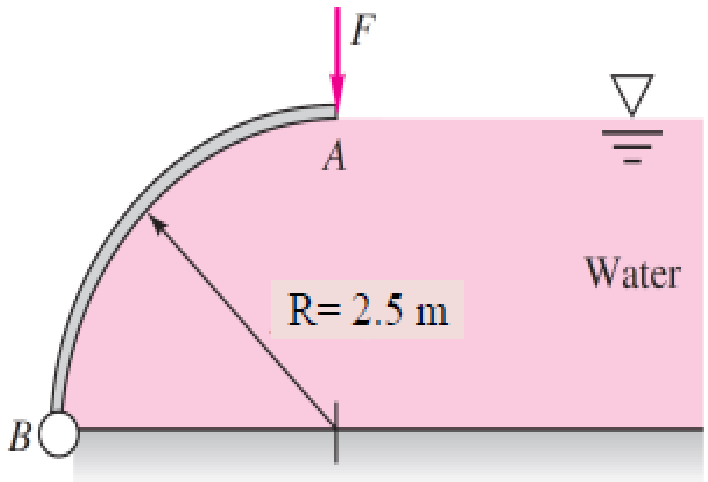
$$\tan\theta = \frac{F_V}{F_H} = \frac{37.9}{36.1} = 1.05 \implies \theta = 46.4°$$

Taking a moment about point **A** at the location of the hinge and equating it to zero gives

$$F_R R \sin\theta - W_{cy} R = 0 \implies W_{cy} = F_R \sin\theta = 52.3 \times \sin 46.4° = 37.9\ kN$$

Example:

Gate AB (as shown in Figure) is a quarter circle with 2.5 m radius and 3 m wide into the paper and hinged at point B. Find the force F just sufficient to keep the gate from opening. The gate is uniform and weighs 13 kN. Take the density of water = 1000 kg/m³.



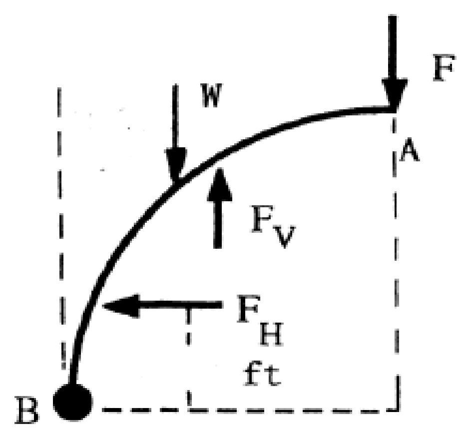
Solution:

The horizontal force is computed for vertical projection of the curved surface AB:

$$F_H = \rho g h_c A = 1000 \times 9.81 \times 1.25 \times (2.5 \times 3)$$
$$= 91968.75\ N$$

Where, F_H acts (2.5 – (2.5/3) = 1.667 m) below point A.

The vertical force equals the weight of the missing piece of water above the gate, as shown below,



$$F_V = (\rho g h_c A)_{Square} - (\rho g h_c A)_{Quarter\ Circle}$$
$$F_V = (\rho g v)_{Square} - (\rho g v)_{Quarter\ Circle}$$
$$F_V = (10^3 \times 9.81 \times 2.5^2 \times 3)_{Square}$$
$$-\left(10^3 \times 9.81 \times \frac{\pi}{4} \times 2.5^2 \times 3\right)_{Quarter}$$
$$F_V = 183937.5 - 144464.2 = 39473.325\ N$$

The line of action **X** for this 39473.3 N force is found by summing moments from above: **ΣMB** = 39473.3 X = 183937.5(1.25) – 144464.2(1.439), X = 0.5583 m

Finally, there is the 13 kN gate weight W, whose centroid is $2R/\pi = 1.592$ m from force F, or (2.5 – 1.592 = 0.91 m) from point B. Then we may sum moments about hinge B to find the force F, using the free-body of the gate:

ΣM_B (clockwise)= 0= F (2.5) + (13000) (0.91) − (39473.3) (0.5583) − (91968.5) (0.833)

F = 34727 N

2.7 Fluids in rigid-body motion

Many fluids such as milk and gasoline are transported in tankers. In an accelerating tanker, the fluid rushes to the back, and some initial splashing occurs. But then a new free surface (usually non-horizontal) is formed, each fluid particle assumes the same acceleration, and the entire fluid moves like a rigid body. No shear stresses develop within the fluid body since there is no deformation and thus no change in shape. Rigid-body motion of a fluid also occurs when the fluid is contained in a tank that rotates about an axis.

Consider a differential rectangular fluid element of side lengths dx, dy, and dz in the x-, y-, and z-directions, respectively, with the z-axis being upward in the vertical direction (see Figure 21). Noting that the differential fluid element behaves like a rigid body, Newton's second law of motion for this element can be expressed as

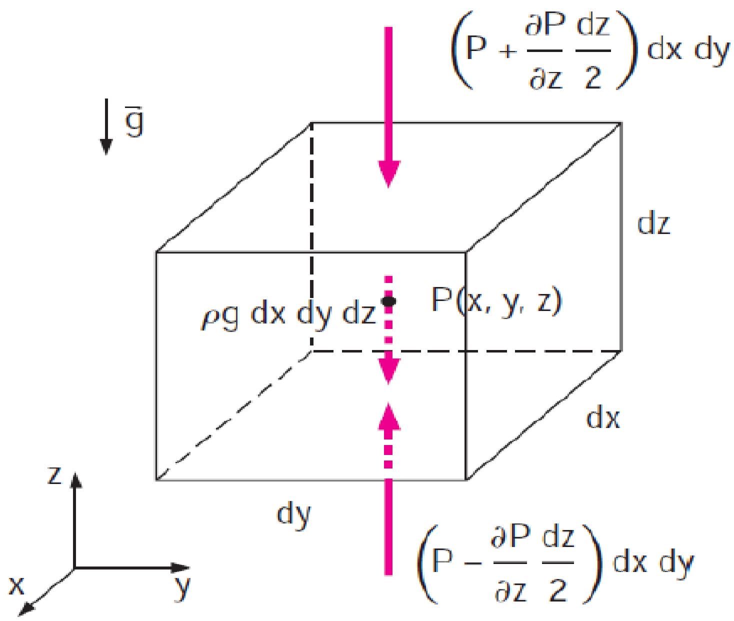


Figure 21. The surface and body forces acting on a differential fluid element in the vertical direction.

where $\boldsymbol{\delta m = \rho dV = \rho\ dx\ dy\ dz}$ is the mass of the fluid element, $\vec{}$ is the acceleration, and$\boldsymbol{\delta F}$ is the net force acting on the element.

Taking the pressure at the center of the element to be **P**, the pressures at the top and bottom surfaces of the element can be expressed as [P + (∂P/∂z) dz/2] and [P - (∂P/∂z) dz/2], respectively. Noting that the pressure force acting on a surface is equal to the average pressure multiplied by the surface area, the net surface force acting on the element in the z-direction is the difference between the pressure forces acting on the bottom and top faces,

$$\delta F_{s,z} = \left(P - \frac{\partial P}{\partial Z}\frac{dz}{2}\right)\ dx\ dy\ - \left(P + \frac{\partial P}{\partial Z}\frac{dz}{2}\right) dx\ dy\ = -\frac{\partial P}{\partial Z} dx\ dy\ dz$$

Similarly, the net surface forces in the x- and y-directions are

$$\delta F_{s,x} = -\frac{\partial P}{\partial x} dx\ dy\ dz \qquad and \qquad \delta F_{s,y} = -\frac{\partial P}{\partial y} dx\ dy\ dz$$

Substituting into Newton's second law of motion

$$\delta F = \delta m \times a = \rho\, dx\, dy\, dz \times a$$

$$\because \; \delta F_{s,x} = -\frac{\partial P}{\partial x} dx\, dy\, dz \;\Rightarrow \therefore \; \rho\, dx\, dy\, dz \times a_x = -\frac{\partial P}{\partial x} dx\, dy\, dz$$

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho \times a_x$, $\frac{\partial P}{\partial y} = -\rho \times a_y$

$$\frac{\partial P}{\partial z} = -\rho \times a_z$$

Special Case 1: Fluids at Rest

For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero, and the relations in previous equation reduce to

Fluids at rest: : $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$, $\frac{\partial P}{\partial z} = 0$

which confirm that, in fluids at rest, the pressure remains constant in any horizontal direction (P is independent of x and y) and varies only in the vertical direction as a result of gravity [and thus P = P(z)]. These relations are applicable for both compressible and incompressible fluids.

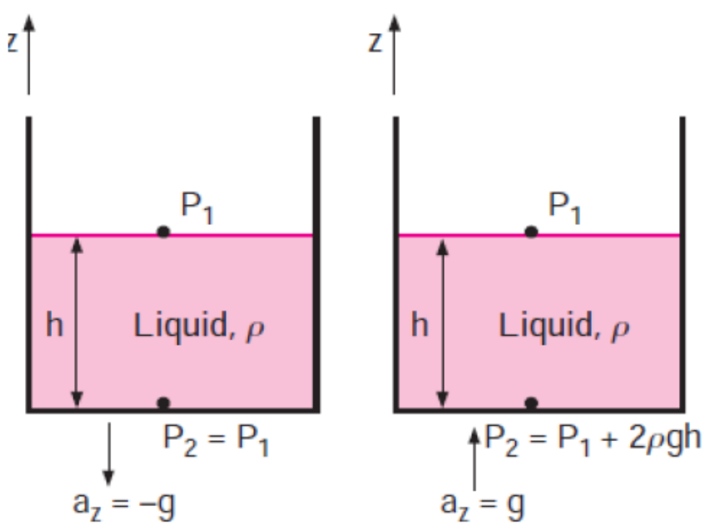
Special Case 2: Free Fall of a Fluid Body

A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero. Therefore, $a_x = a_y = 0$ and $a_z = -g$. Then the equations of motion for accelerating fluids reduce to

Free-falling fluids: $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ P= Constant

Therefore, in a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout. (Actually, the gage pressure is slightly above zero due to surface tension, which holds the drop intact).

When the direction of motion is reversed and the fluid is forced to accelerate vertically with [a_z = + g] by placing the fluid container in an elevator or a space vehicle propelled upward by a rocket engine, the pressure gradient in the z-direction is $\frac{\partial y}{\partial x} = -2\rho g$. Therefore, the pressure difference across a fluid layer now **doubles** relative to the stationary fluid case as shown inFigure 22.



(a) Free fall of a liquid (b)Upward acceleration of a liquid with az = +g

Figure 22. The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

2.8 Acceleration on a Straight Path

Consider a container partially filled with a liquid. The container is moving on a straight path with a constant acceleration. We take the projection of the path of motion on the horizontal plane to be the x-axis, and the projection on the vertical plane to be the z-axis, as shown in Figure 2.28. The x- and z-components of acceleration are ax and az. There is no movement in the y-direction, and thus the acceleration in that direction is zero, ay = 0. Then the equations of motion for accelerating fluids reduce to

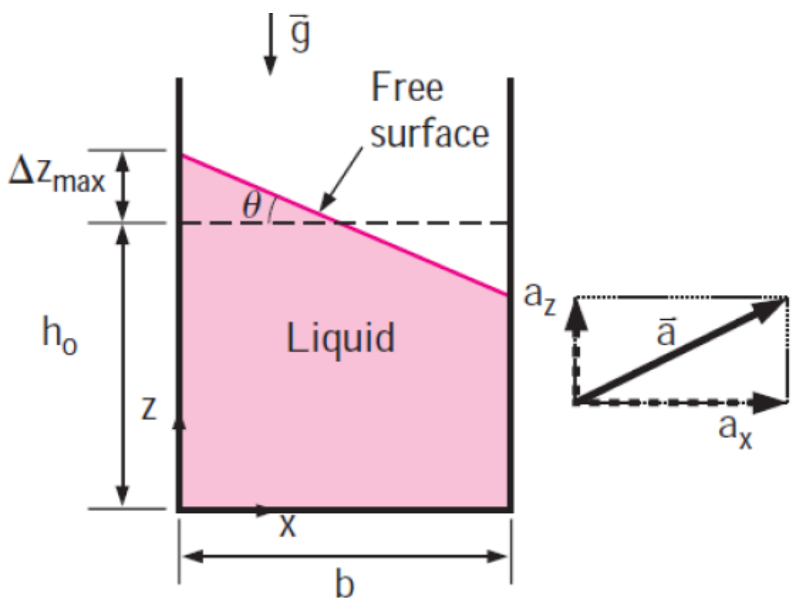


Figure 23. Rigid-body motion of a liquid in a linearly accelerating tank.

: $\frac{\partial P}{\partial x} = -\rho \times a_x$, $\frac{\partial P}{\partial y} = 0$ $\frac{\partial P}{\partial z} = -\rho\,(g + a_z)$

Therefore, pressure is independent of y. Then the total differential of P = P(x, z), which is $\left(\frac{\partial P}{\partial x}\right) dx + \left(\frac{\partial P}{\partial z}\right) dz$, becomes

$dP = -\rho a_x dx - \rho(g + a_z)dz$. For ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integration to be

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

Taking point 1 to be the origin (x = 0, z = 0) where the pressure is Po and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as,

Pressure variation:

$$P = P_o - \rho \times a_x - \rho(g + a_z)z$$

The vertical rise (or drop) of the free surface at point 2 relative to point 1 can be determined by choosing both 1 and 2 on the free surface (so that $P_1 = P_2$), and solving Equation

$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$ for $(z_2 - z_1)$ as shown Figure 24

$$\text{Vertical rise of surface:} \Delta z_s = \Delta z_{s2} - \Delta z_{s1} = -\frac{a_x}{g+a_z}(x_2 - x_1)$$

where z_s is the z-coordinate of the liquid's free surface. The equation for surfaces of constant pressure, called isobars, is obtained from $\boldsymbol{dP = -\rho a_x dx - \rho(g + a_z)dz}$ by setting dP = 0 and replacing z by z_{isobar}, which is the z-coordinate (the vertical distance) of the surface as a function of x. It gives

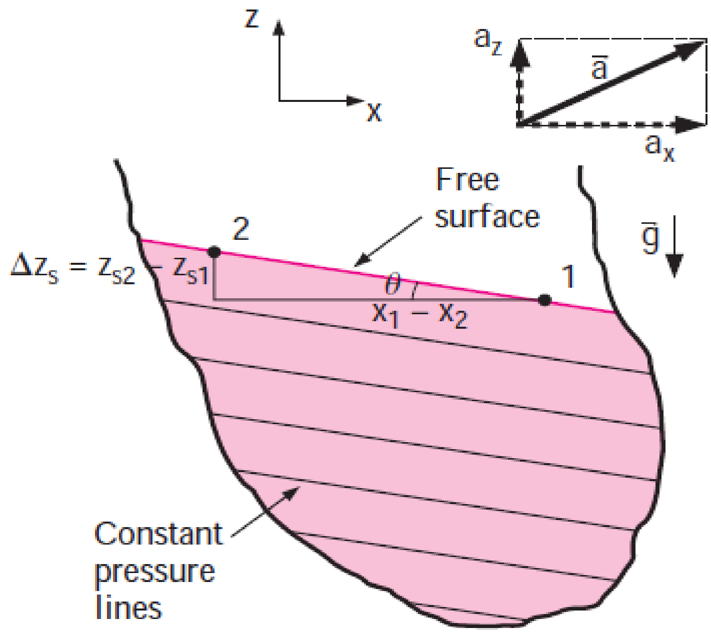


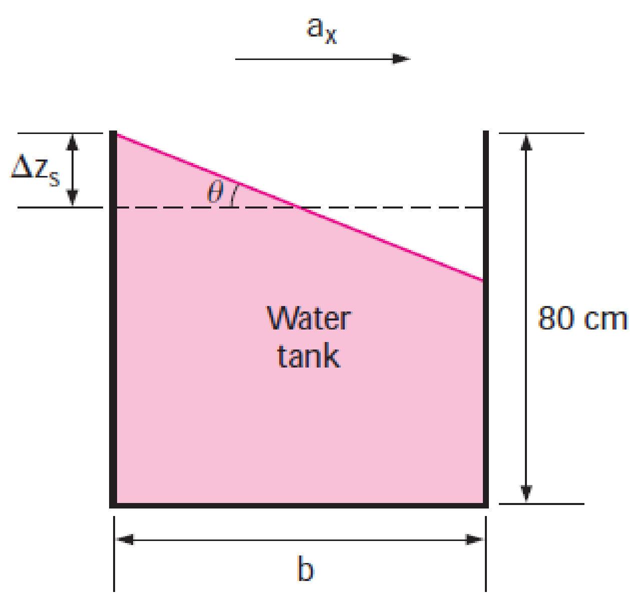
Figure 24. Lines of constant pressure (which are the projections of the surfaces of constant pressure on the xz - plane) in a linearly accelerating liquid, and the vertical rise.

Example:

An 80-cm-high fish tank of cross section 2 m × 0.6 m that is initially filled with water is to be transported on the back of a truck (see Figure). The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?

Solution:

We take the x-axis to be the direction of motion, the z-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is



$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90000 - 0)}{10 \times 3600} = 2.5\ m/s^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \Rightarrow \quad \theta = 14.3°$$

The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical mid-plane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the mid-plane for the two possible orientations becomes

Case 1: the long side is parallel to the direction of motion:

$$\tan\theta = \frac{\Delta z_{s1}}{(b_1/2)} \Longrightarrow \Delta z_{s1} = (b_1/2) \times \tan\theta = [2/2] \times 0.255 = 0.255\ m$$

$$= 25.5\ cm$$

Case 2: the short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2) \times \tan\theta = [0.6/2] \times 0.255 = 0.076\ m = 7.6\ cm$$