Stress, Strain and Elastic Moduli

Consider a solid rod with length L_o, diameter D_o and cross sectional area A_o as depicted in Figure below (a). When tensile forces, those stretching the solid, are applied to this specimen, the rod becomes extended, or stretched as shown in Figure below (b). At the same time it becomes narrower. Its new length, diameter and cross sectional area are ***L*, *D*** and ***A***. The effects of the applied load can only be meaningfully compared between samples if we compare the force experienced per unit cross sectional area of the sample. In other words, we need to compare forces experienced by the same area on different samples. We define ***engineering tensile stress*** as the applied force per unit original cross sectional area of the sample where the force is perpendicular to the area,

$$\sigma = \frac{F}{A_o}$$

Tensile stresses pull away the faces of application. The forces are at right angles to the faces. If the applied forces are pushing the faces inwards, compressing the sample, then the stresses are said to be ***compressive.*** The actual stress, or instantaneous stress, is the force per instantaneous unit area. If during the application of the force ***F*** the cross sectional area is ***A*,** then ***true stress*** σ_t is defined as,

$$\sigma_t = \frac{F}{A}$$

When comparing the amounts of extension different samples exhibit under a given tensile stress, it is useful to compare the extensions per unit original length. If $\Delta L = L - L_o$ is the extension under an applied tensile load then ***engineering strain*** is defined as,

$$\varepsilon = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o}$$

Engineering strain is based on extensions with respect to the original length L_o. Consider what happens when we increase the tensile load ***F*** by a small amount δF as shown in Figure below (c). The length *L* changes by δL. The instantaneous incremental increase in length per unit length $\delta L/L$ is defined as incremental true strain; True strain is then the summation of all incremental true strains from original length, L_o, to final length ***L***,

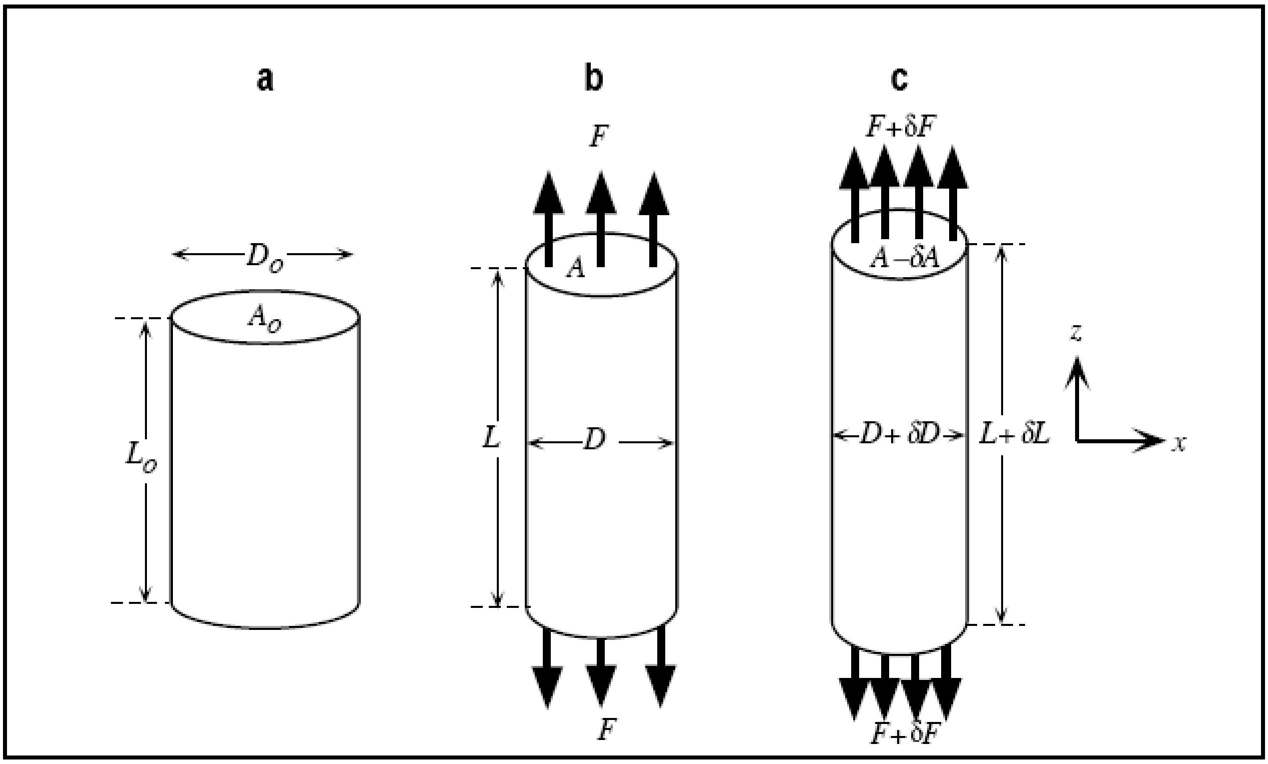
$$\varepsilon_t = \ln(1 + \varepsilon)$$

An elastic behavior in which the strain exhibited by a body increases with the applied stress. If the stress is removed, the strain returns to **zero**. This type of elastic behavior is called **Hooke's law**. The proportionality is usually written as

$$\sigma = E\varepsilon$$

Where ***E*** is an elastic modulus that is called **Young's modulus**. Young's modulus depends on the nature of bonding between the atoms or molecules in the solid in as much as the strain is a result of the stretching of bonds between atoms. A solid that has been strained by an applied stress is said to be ***elastically deformed*** if it returns to its original shape after the removal of the stress. By convention, a positive stress pulls, or creates tension, and a negative stress pushes, or creates compression. Thus, compression results in a negative strain or compression as we expect. When a solid is extended along the direction of an applied tensile stress, along ***z***, it becomes narrower in the perpendicular directions. The longitudinal stress, σz, therefore induces not only a longitudinal strain**,** εz**,** but also a lateral strain, εx, along ***x*** in a direction perpendicular to ***z*.** The lateral strain along ***x*,** along any direction perpendicular to ***z***, is defined as $\varepsilon x = \Delta D / D_o = (D - D_o) / D_o$ which is a negative quantity for tensile stresses and positive for compression stresses. The ratio of lateral to longitudinal strain is defined as the ***Poisson's ratio*,**υ **,** for that material,

$$\upsilon = -\frac{\varepsilon_x}{\varepsilon_z} = -\frac{\text{Lateral strain}}{\text{Longitudinal strength}}$$



There are other forms of stresses and strains in engineering besides those tensile stresses. When opposite forces act tangentially on opposite faces of a brick-like solid as in Figure below, the resulting stresses are called **shear stresses** as they tend to shear the solid. ***Shear stress***, denoted byτ, is defined as the tangential force per unit surface area on which it acts,

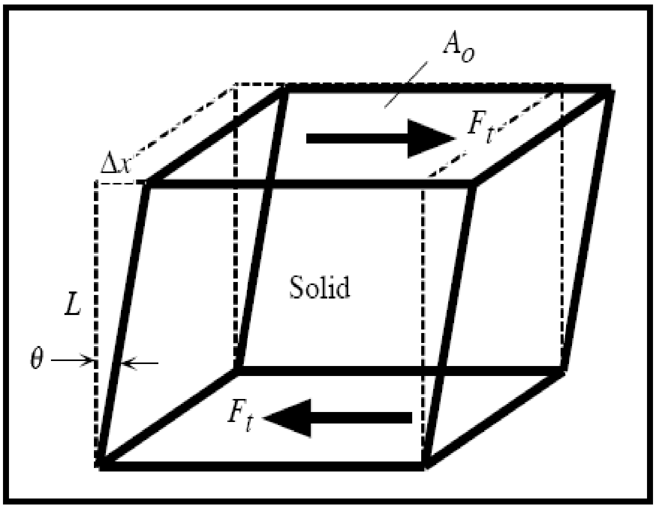
$$\tau = \frac{F_t}{A}$$

Where F_t is the tangential force acting on surface area ***A***. Shear stresses cause a solid to become sheared or skewed. In Figure below, shear stresses acting on a brick-type solid induce the opposite surfaces, separated by distance ***L***, to be relatively displaced by Δx along the forces (along ***x***). Greater is Δx with respect to ***L***, bigger is the extent of skewing. Shear strain,Υ, is defined by

$$\gamma = \frac{\Delta x}{L} = \tan(\theta) \approx \theta$$

Where ***θ*** is the shearing angle. As typically ***θ*** is small, tan ***θ*** ≈ ***θ*** and shear strain is simply this angle of shear. All experiments indicate that for a given material and for small shear strains, the shear deformation is elastic and obeys Hooke's law. ***G*** called the shear modulus (also known as modulus of rigidity) relates the shear stress and strain in elastic shear deformation via

$$\tau = G\gamma$$



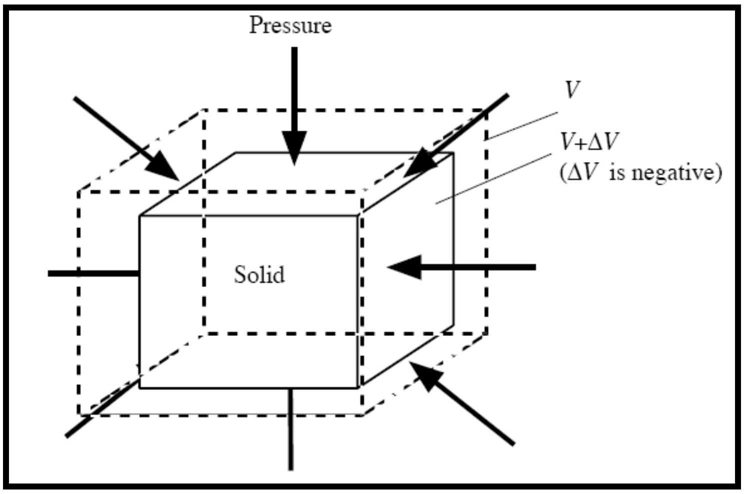
Another form of stress arises by the uniform application of forces over the whole surface of the body as, for example, when the body is immersed deep into ocean. Force per unit area in this case is called **pressure**, ***P***. Even though pressure is always compressing a body, it is nonetheless, denoted as a positive quantity. Suppose that an applied pressure changes the volume by an amount **Δ*V*** as shown in Figure below. Then ***volume strain***, **Δ**, is defined by

$$\Delta = \frac{\Delta V}{V}$$

For small volume strains, the volume deformation is elastic and obeys Hooke's law with an elastic modulus that is called the ***bulk modulus***, ***K***:

$$P = -K\Delta$$

The negative sign in Equation therefore ensures that the bulk modulus, which is a material property, is tabulated as a positive quantity. Elastic deformation is said to occur ***instantaneously.*** When a stress is applied to a body, the bonds between atoms become stretched or compressed almost immediately. Furthermore, when the stress is removed, the solid instantaneously and the induced strain occurs returns to its original shape and size.



Types of Elastic Deformation

Stress	Strain	Elastic Modulus	Hooke's Law
Tensile stress, σ	Tensile strain, $\varepsilon = \Delta L / L_o$	Young's Modulus, E	$\sigma = E\varepsilon$
Shear stress, τ	Shear strain, $\gamma = \theta$	Shear Modulus, G	$\tau = G\gamma$
Hydrostatic Pressure, P	Volume strain, $\Delta = \Delta V / V$	Bulk Modulus, K	$P = -K\Delta$

	E	***G***	***K***
E		$E = 2G(1+\upsilon)$	$E = 3K(1-2\upsilon)$
G	$G = \dfrac{E}{2(1+\upsilon)}$		$G = \dfrac{3(1-2\upsilon)}{2(1+\upsilon)}K$
K	$K = \dfrac{E}{3(1-2\upsilon)}$	$K = \dfrac{2(1+\upsilon)}{3(1-2\upsilon)}G$	