الكلية	التربية للعلوم الصرفة
القسم	الرياضيات
المادة باللغة الانجليزية	functional analysis
المادة باللغة العربية	التحليل الدالي
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2.2 Open Mappings and Closed Graphs

2.2.1 The Open Mapping Theorem

A map $f : X \to Y$ between topological spaces is called **open** if the image of every open subset of X under f is an open subset of Y.

Theorem 2.8 (Open Mapping Theorem). *Let X, Y be Banach spaces and let $A : X \to Y$ be a surjective bounded linear operator. Then A is open.*

Proof. See page 64. □

The key step in the proof of Theorem 2.8 is the next lemma, which asserts that the closure $\overline{A(B)}$ of the image of the open unit ball $B \subset X$ under a surjective bounded linear operator $A : X \to Y$ contains an open ball in Y centered at the origin. Its proof relies on the Baire Category Theorem 1.55. Lemma 2.10 below asserts that if an open ball in Y centered at the origin is contained in $\overline{A(B)}$ then it is contained in $A(B)$.

Lemma 2.9. *Let X, Y, and A be as in Theorem 2.8. Then there exists a constant $\delta > 0$ such that*

$$\left\{ y \in Y \,\middle|\, \|y\|_Y < \delta \right\} \subset \overline{\{Ax \,|\, x \in X, \, \|x\|_X < 1\}}. \tag{2.6}$$

Proof. For $C \subset Y$ and $\lambda > 0$ define $\lambda C := \{\lambda y \,|\, y \in C\}$. Consider the sets

$$B := \{x \in X \,|\, \|x\|_X < 1\}, \qquad C := A(B) = \{Ax \,|\, x \in X, \, \|x\|_X < 1\}.$$

Then $X = \bigcup_{n \in \mathbb{N}} nB$ and so $Y = \bigcup_{n \in \mathbb{N}} A(nB) = \bigcup_{n \in \mathbb{N}} nC$ because A is surjective. Since Y is complete, at least one of the sets nC is not nowhere dense, by the Baire Category Theorem 1.55. Hence the set $\overline{nC}$ has a nonempty interior for some $n \in \mathbb{N}$ and this implies that the set $\overline{2^{-1}C}$ has a nonempty interior. Choose $y_0 \in Y$ and $\delta > 0$ such that

$$B_\delta(y_0) \subset \overline{2^{-1}C}.$$

We claim that (2.6) holds with this constant δ. To see this, fix an element $y \in Y$ such that $\|y\|_Y < \delta$. Then $y_0 + y \in \overline{2^{-1}C}$ and $y_0 \in \overline{2^{-1}C}$. Hence there exist sequences $x_i, x_i' \in 2^{-1}B$ such that

$$y_0 + y = \lim_{i \to \infty} Ax_i', \qquad y_0 = \lim_{i \to \infty} Ax_i.$$

Hence $x_i' - x_i \in B$, so $A(x_i' - x_i) \in C$, and $y = \lim_{i \to \infty} A(x_i' - x_i) \in \overline{C}$. Thus (2.6) holds as claimed. This proves Lemma 2.9. □

Lemma 2.10. *Let X and Y be Banach spaces and let $A : X \to Y$ be a bounded linear operator. If $\delta > 0$ and*

$$\left\{y \in Y \,\middle|\, \|y\|_Y < \delta\right\} \subset \overline{\left\{Ax \,\middle|\, x \in X,\, \|x\|_X < 1\right\}}, \tag{2.7}$$

then

$$\left\{y \in Y \,\middle|\, \|y\|_Y < \delta\right\} \subset \left\{Ax \,\middle|\, x \in X,\, \|x\|_X < 1\right\}. \tag{2.8}$$

Proof. The proof is based on the following observation.

Claim. *Let $y \in Y$ such that $\|y\|_Y < \delta$. Then there exists a sequence $(x_k)_{k \in \mathbb{N}_0}$ in X such that*

$$\begin{aligned} &\|x_0\|_X < \frac{\|y\|_Y}{\delta}, \quad \|x_k\|_X < \frac{\delta - \|y\|_Y}{\delta 2^k} \quad \text{for } k = 1, 2, 3, \ldots, \\ &\|y - Ax_0 - \cdots - Ax_k\|_Y < \frac{\delta - \|y\|_Y}{2^{k+1}} \quad \text{for } k = 0, 1, 2, \ldots. \end{aligned} \tag{2.9}$$

We prove the claim by an induction argument. By (2.7) the closed ball of radius δ in Y is contained in the closure of the image under A of the open ball of radius one in X. Hence every nonzero vector $y \in Y$ satisfies

$$y \in \overline{\left\{Ax \,\middle|\, x \in X,\, \|x\|_X < \delta^{-1} \|y\|_Y\right\}}. \tag{2.10}$$

Fix an element $y \in Y$ such that $\|y\|_Y < \delta$ and define $\varepsilon := \delta - \|y\|_Y > 0$. Then, by (2.10), there exists a vector $x_0 \in X$ such that $\|x_0\|_X < \delta^{-1} \|y\|_Y$ and $\|y - Ax_0\|_Y < \varepsilon 2^{-1}$. Use (2.10) again with y replaced by $y - Ax_0$ to find a vector $x_1 \in X$ such that $\|x_1\|_X < \varepsilon\delta^{-1}2^{-1}$ and $\|y - Ax_0 - Ax_1\|_Y < \varepsilon 2^{-2}$. Once the vectors $x_0, \ldots, x_k$ have been found such that (2.9) holds, we have $\|y - \sum_{i=0}^k Ax_i\|_Y < \varepsilon 2^{-k-1}$ and so, by (2.10), there is a vector $x_{k+1} \in X$ such that $\|x_{k+1}\|_X < \varepsilon\delta^{-1}2^{-k-1}$ and $\|y - \sum_{i=0}^k Ax_i - Ax_{k+1}\|_Y < \varepsilon 2^{-k-2}$. Hence the existence of a sequence $(x_k)_{k \in \mathbb{N}_0}$ in X that satisfies (2.9) follows from the axiom of dependent choice (see page 10). This proves the claim.

Now fix an element $y \in Y$ such that $\|y\|_Y < \delta$. By the claim, there is a sequence $(x_k)_{k \in \mathbb{N}_0}$ in X that satisfies (2.9) and hence $\sum_{k=0}^\infty \|x_k\|_X < 1$. It then follows from Lemma 1.45 that the limit $x := \sum_{k=0}^\infty x_k = \lim_{k\to\infty} \sum_{i=0}^k x_i$ exists. This limit satisfies the inequality $\|x\|_X \le \sum_{k=0}^\infty \|x_k\|_X < 1$ as well as $Ax = \lim_{k\to\infty} \sum_{i=0}^k Ax_i = y$. Here the last equation follows from (2.9). This proves the inclusion (2.8) and Lemma 2.10. □

Proof of Theorem 2.8. Let $\delta > 0$ be the constant of Lemma 2.9 and denote by $B \subset X$ be the open unit ball. Then $B_\delta(0; Y) \subset \overline{A(B)}$ by Lemma 2.9 and hence $B_\delta(0; Y) \subset A(B)$ by Lemma 2.10.

Now fix an open set $U \subset X$. Let $y_0 \in A(U)$ and choose $x_0 \in U$ such that $Ax_0 = y_0$. Since U is open there is an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$. We prove that $B_{\delta\varepsilon}(y_0) \subset A(U)$. Choose $y \in Y$ such that $\|y - y_0\|_Y < \delta\varepsilon$. Then $\|\varepsilon^{-1}(y - y_0)\|_Y < \delta$ and hence there exists an element $\xi \in X$ such that

$$\|\xi\|_X < 1, \qquad A\xi = \varepsilon^{-1}(y - y_0).$$

This implies $y = y_0 + \varepsilon A\xi = A(x_0 + \varepsilon\xi) \in A(U)$, because $x_0 + \varepsilon\xi \in B_\varepsilon(x_0) \subset U$. Thus we have proved that, for every $y_0 \in A(U)$, there exists a number $\varepsilon > 0$ such that $B_{\delta\varepsilon}(y_0) \subset A(U)$. Hence $A(U)$ is an open subset of Y and this proves Theorem 2.8. □

If $A : X \to Y$ is a surjective bounded linear operator between Banach spaces, then it descends to a bijective bounded linear operator from the quotient space $X/\ker(A)$ to Y (see Theorem 1.29). The next corollary asserts that the induced operator $\overline{A} : X/\ker(A) \to Y$ has a bounded inverse whose norm is bounded above by δ^{-1}, where the constant $\delta > 0$ is as in Lemma 2.9.

Corollary 2.11. *Let X, Y, and A be as in Theorem 2.8 and let $\delta > 0$ be the constant of Lemma 2.9. Then*

$$\inf_{\substack{x \in X \\ Ax = y}} \|x\|_X \le \delta^{-1} \|y\|_Y \qquad \text{for all } y \in Y. \tag{2.11}$$

Proof. Let $y \in Y$ and choose a constant $c > \delta^{-1}\|y\|_Y$. Then $\|c^{-1}y\|_Y < \delta$ and so, by Lemma 2.9 and Lemma 2.10, there exists an element $\xi \in X$ such that $A\xi = c^{-1}y$ and $\|\xi\|_X < 1$. Hence $x := c\xi$ satisfies $\|x\|_X = c\|\xi\|_X < c$ and $Ax = cA\xi = y$. This proves (2.11) and Corollary 2.11. □

An important consequence of the open mapping theorem is the special case of Corollary 2.11 where A is bijective.

Theorem 2.12 (Inverse Operator Theorem). *Let X and Y be Banach spaces and let $A : X \to Y$ be a bijective bounded linear operator. Then the inverse operator $A^{-1} : Y \to X$ is bounded.*

Proof. By Theorem 2.8 the linear operator $A : X \to Y$ is open. Hence its inverse is continuous and is therefore bounded by Theorem 1.17. Alternatively, use Corollary 2.11 to deduce that $\|A^{-1}\| \le \delta^{-1}$, where $\delta > 0$ is the constant of Lemma 2.9. □

Corollary 2.16. *Let X be a Banach space and let $X_1, X_2 \subset X$ be two closed linear subspaces such that $X = X_1 \oplus X_2$, i.e. $X_1 \cap X_2 = \{0\}$ and every vector $x \in X$ can be written as $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Then there exists a constant $c \geq 0$ such that*

$$\|x_1\| + \|x_2\| \leq c\,\|x_1 + x_2\| \tag{2.12}$$

for all $x_1 \in X_1$ and all $x_2 \in X_2$.

Proof. The vector space $X_1 \times X_2$ is a Banach space with the norm function

$$X_1 \times X_2 \to [0, \infty) : (x_1, x_2) \mapsto \|(x_1, x_2)\| := \|x_1\| + \|x_2\|$$

(see Exercise 1.30) and the linear operator $A : X_1 \times X_2 \to X$, defined by $A(x_1, x_2) := x_1 + x_2$ for $(x_1, x_2) \in X_1 \times X_2$, is bijective by assumption and bounded by the triangle inequality. Hence its inverse is bounded by the Inverse Operator Theorem 2.12. This proves Corollary 2.16. □

Example 2.13. This example shows that the hypothesis that X and Y are complete cannot be removed in Theorems 2.8 and 2.12. As in Example 2.6, let $X \subset \ell^\infty$ be the subspace of sequences $x = (x_k)_{k\in\mathbb{N}}$ of real numbers that vanish for sufficiently large k, equipped with the supremum norm. Thus X is a normed vector space but is not a Banach space. Define the operator $A : X \to X$ by $Ax := (k^{-1}x_k)_{k\in\mathbb{N}}$ for $x = (x_k)_{k\in\mathbb{N}} \in X$. Then A is a bijective bounded linear operator but its inverse is unbounded.

Example 2.14. Here is another example where X is complete and Y is not. Let $X = Y = C([0, 1])$ be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ equipped with the norms

$$\|f\|_X := \sup_{0\le t\le 1} |f(t)|, \qquad \|f\|_Y := \sqrt{\int_0^1 |f(t)|^2\,dt}$$

Then X is a Banach space, Y is a normed vector space, and the identity map $A = \mathrm{id} : X \to Y$ is a bijective bounded linear operator with an unbounded inverse.

Example 2.15. Here is an example where Y is complete and X is not. This example requires the axiom of choice. Let Y be an infinite-dimensional Banach space and choose an unbounded linear functional $\Phi : Y \to \mathbb{R}$. The existence of such a linear functional is shown in part (iv) of Example 1.25 and its kernel is a dense linear subspace of Y by Exercise 1.38. Define the normed vector space $(X, \|\cdot\|_X)$ by

$$X := \{(x, t) \in Y \times \mathbb{R} \mid \Phi(x) = 0\}, \qquad \|(x, t)\|_X := \|x\|_Y + |t|$$

for $(x, t) \in X$. Then X is not complete. Choose a vector $y_0 \in Y$ such that $\Phi(y_0) = 1$ and define the linear map $A : X \to Y$ by

$$A(x, t) := x + ty_0 \qquad \text{for } (x, t) \in X.$$

Then A is a bijective bounded linear operator. Its inverse is given by

$$A^{-1}y = (y - \Phi(y)y_0, \Phi(y))$$

for $y \in H$ and hence is unbounded.

Example 2.15 relies on a decomposition of a Banach space as a direct sum of two linear subspaces where one of them is closed and the other is dense. The next corollary establishes an important estimate for a pair of *closed* subspaces of a Banach space X whose **direct sum** is equal to X.