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عنوان المحاضرة باللغة الانجليزية	Uniform Boundedness
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2.1 Uniform Boundedness

Let X be a set. A family $\{f_i\}_{i\in I}$ of functions $f_i : X \to Y_i$, indexed by a set I and each taking values in a normed vector space Y_i, is called **pointwise bounded**, if

$$\sup_{i\in I} \|f_i(x)\|_{Y_i} < \infty \qquad \text{for all } x \in X. \tag{2.1}$$

Theorem 2.1 (Uniform Boundedness). *Let X be a Banach space, let I be any set, and, for each $i \in I$, let let Y_i be a normed vector space and let $A_i : X \to Y_i$ be a bounded linear operator. Assume that the operator family $\{A_i\}_{i\in I}$ is pointwise bounded. Then $\sup_{i\in I} \|A_i\| < \infty$.*

Lemma 2.2. *Let (X, d) be a nonempty complete metric space, let I be any set, and, for each $i \in I$, let $f_i : X \to \mathbb{R}$ be a continuous function. Assume that the family $\{f_i\}_{i\in I}$ is pointwise bounded. Then there exists a point $x_0 \in X$ and a number $\varepsilon > 0$ such that*

$$\sup_{i\in I} \sup_{x\in B_\varepsilon(x_0)} |f_i(x)| < \infty.$$

Proof. For $n \in \mathbb{N}$ and $i \in I$ define the set

$$F_{n,i} := \left\{x \in X \,\middle|\, |f_i(x)| < n\right\}.$$

This set is closed because f_i is continuous. Hence the set

$$F_n := \bigcap_{i\in I} F_{n,i} = \left\{x \in X \,\middle|\, \sup_{i\in I}|f_i(x)| \le n\right\}$$

is closed for every $n \in \mathbb{N}$. Moreover,

$$X = \bigcup_{n\in\mathbb{N}} F_n,$$

because the family $\{f_i\}_{i\in I}$ is pointwise bounded. Since (X, d) is a nonempty complete metric space, it follows from the Baire Category Theorem 1.55 that the sets F_n cannot all be nowhere dense. Since these sets are all closed, there exists an integer $n \in \mathbb{N}$ such that F_n has nonempty interior. Hence there exists an integer $n \in \mathbb{N}$, a point $x_0 \in X$, and a number $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset F_n$. Hence $\sup_{i\in I} \sup_{x\in B_\varepsilon(x_0)}|f_i(x)| \le n$ and this proves Lemma 2.2.

Remark 2.3. The above argument in the proof of Theorem 2.1, which asserts that (2.2) implies (2.3), can be rewritten as the inequality

$$\sup_{\substack{x\in X\\ \|x-x_0\|_X<\varepsilon}} \|Ax\|_Y \ge \varepsilon \|A\| \tag{2.4}$$

for all $A \in \mathcal{L}(X,Y)$, all $x_0 \in X$, and all $\varepsilon > 0$. With this understood, one can prove the Uniform Boundedness Theorem as follows (see Sokal [57]). Let $\{A_i\}_{i\in I}$ be a sequence of bounded linear operators $A_i : X \to Y_i$ such that $\sup_{i\in I} \|A_i\| = \infty$. Then the axiom of countable choice asserts that there is a sequence $i_n \in I$ such that $\|A_{i_n}\| \geq 4^n$ for all $n \in \mathbb{N}$. Now use the axiom of dependent choice, and the estimate (2.4) with $A = A_{i_n}$ and $\varepsilon = 1/3^n$, to find a sequence $x_n \in X$ such that, for all $n \in \mathbb{N}$,

$$\|x_n - x_{n-1}\|_X \leq \frac{1}{3^n}, \qquad \|A_{i_n} x_n\|_{Y_i} \geq \frac{2}{3}\frac{1}{3^n}\|A_{i_n}\|.$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges to an element $x^* \in X$ such that $\|x^* - x_n\|_X \leq \frac{1}{2}\frac{1}{3^n}$. Thus $\|A_{i_n}x^*\|_{Y_i} \geq (\frac{2}{3} - \frac{1}{2})\frac{1}{3^n}\|A_{i_n}\| \geq \frac{1}{6}(\frac{4}{3})^n$ for all $n \in \mathbb{N}$ and so the operator family $\{A_i\}_{i\in I}$ is not pointwise bounded. This argument circumvents the Baire Category Theorem.

The Uniform Boundedness Theorem is also known as the **Banach–Steinhaus Theorem**. A useful consequence is that the limit of a pointwise convergent sequence of bounded linear operators is again a bounded linear operator. This is the content of Theorem 2.5 below.

Definition 2.4. *Let X and Y be normed vector spaces. A sequence of bounded linear operators $A_i : X \to Y$, $i \in \mathbb{N}$, is said to* **converge strongly** *to a bounded linear operator $A : X \to Y$ if $Ax = \lim_{i\to\infty} A_i x$ for all $x \in X$.*

Theorem 2.5 (Banach–Steinhaus). *Let X and Y be Banach spaces and let $A_i : X \to Y$, $i \in \mathbb{N}$, be a sequence of bounded linear operators. Then the following are equivalent.*

(i) *The sequence $(A_i x)_{i\in\mathbb{N}}$ converges in Y for every $x \in X$.*

(ii) *$\sup_{i\in\mathbb{N}} \|A_i\| < \infty$ and there is a dense subset $D \subset X$ such that $(A_i x)_{i\in\mathbb{N}}$ is a Cauchy sequence in Y for every $x \in D$.*

(iii) *$\sup_{i\in\mathbb{N}} \|A_i\| < \infty$ and there is a bounded linear operator $A : X \to Y$ such that A_i converges strongly to A and $\|A\| \leq \liminf_{i\to\infty} \|A_i\|$.*

The equivalence of (i) and (iii) continues to hold when Y is not complete. The equivalence of (ii) and (iii) continues to hold when X is not complete.

Proof. That (iii) implies both (i) and (ii) is obvious.

We prove that (i) implies (iii). Since convergent sequences are bounded, the sequence $(A_i)_{i\in\mathbb{N}}$ is pointwise bounded. Since X is complete it follows from Theorem 2.1 that $\sup_{i\in\mathbb{N}} \|A_i\| < \infty$. Define the map $A : X \to Y$ by $Ax := \lim_{i\to\infty} A_i x$ for $x \in X$. This map is linear and

$$\|Ax\|_Y = \lim_{i\to\infty} \|A_i x\|_Y = \liminf_{i\to\infty} \|A_i x\|_Y \leq \liminf_{i\to\infty} \|A_i\| \, \|x\|_X \qquad (2.5)$$

for all $x \in X$. Hence A is bounded and $\|A\| \leq \liminf_{i\to\infty} \|A_i\| < \infty$.

We prove that (ii) implies (iii). Define $c := \sup_{i\in\mathbb{N}} \|A_i\| < \infty$. Let $x \in X$ and $\varepsilon > 0$. Choose $\xi \in D$ such that $c\,\|x - \xi\| < \frac{\varepsilon}{3}$. Since $(A_i\xi)_{i\in\mathbb{N}}$ is a Cauchy sequence, there exists an integer $n_0 \in \mathbb{N}$ such that $\|A_i\xi - A_j\xi\|_Y < \frac{\varepsilon}{3}$ for all $i, j \in \mathbb{N}$ with $i, j \geq n_0$. This implies

$$\begin{aligned}
\|A_ix - A_jx\|_Y &\leq \|A_ix - A_i\xi\|_Y + \|A_i\xi - A_j\xi\|_Y + \|A_j\xi - A_jx\|_Y \\
&\leq \|A_i\|\,\|x - \xi\|_X + \|A_i\xi - A_j\xi\|_Y + \|A_j\|\,\|\xi - x\|_X \\
&\leq 2c\,\|x - \xi\|_X + \|A_i\xi - A_j\xi\|_Y < \tfrac{2\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for all $i, j \in \mathbb{N}$ with $i, j \geq n_0$. Hence $(A_ix)_{i\in\mathbb{N}}$ is a Cauchy sequence and so it converges because Y is complete. The limit operator A satisfies (2.5) and this proves Theorem 2.5.

Example 2.6. This example shows that the hypothesis that X is complete cannot be removed in Theorems 2.1 and 2.5. Consider the space

$$X := \left\{x = (x_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \,|\, \exists n \in \mathbb{N}\,\forall i \in \mathbb{N} : i \geq n \implies x_i = 0\right\}$$

with the supremum norm $\|x\| := \sup_{i\in\mathbb{N}}|x_i|$. This is a normed vector space. It is not complete, but is a linear subspace of ℓ^∞ whose closure $\overline{X} = c_0$ is the subspace of sequences of real numbers that converge to zero. Define the linear operators $A_n : X \to X$ and $A : X \to X$ by

$$A_nx := (x_1, 2x_2, \ldots, nx_n, 0, 0, \cdots), \qquad Ax := (ix_i)_{i\in\mathbb{N}}$$

for $n \in \mathbb{N}$ and $x = (x_i)_{i\in\mathbb{N}} \in X$. Then $Ax = \lim_{n\to\infty} A_nx$ for every $x \in X$ and $\|A_n\| = n$ for every $n \in \mathbb{N}$. Thus the sequence $\{A_nx\}_{n\in\mathbb{N}}$ is bounded for every $x \in X$, the linear operator A is not bounded, and the sequence A_n converges strongly to A.

Corollary 2.7 (Bilinear Map). *Let X be a Banach space and let Y and Z be normed vector spaces (over $\mathbb{R}$ or $\mathbb{C}$). Let $B : X \times Y \to Z$ be a bilinear map. Then the following are equivalent.*

(i) *B is* **bounded**, *i.e. there is a constant $c \geq 0$ such that*

$$\|B(x,y)\|_Z \leq c\,\|x\|_X\,\|y\|_Y$$

for all $x \in X$ and all $y \in Y$.

(ii) *B is continuous.*

(iii) *For every $x \in X$ the linear map $Y \to Z : y \mapsto B(x,y)$ is continuous and, for every $y \in Y$, the linear map $X \to Z : x \mapsto B(x,y)$ is continuous.*

Proof. If (i) holds then B is locally Lipshitz continuous and hence is continuous. Thus (i) implies (ii). That (ii) implies (iii) is obvious. We prove that (iii) implies (i). Thus assume (iii), define

$$S := \{y \in Y \mid \|y\|_Y = 1\},$$

and, for $y \in S$, define the linear operator $A_y : X \to Z$ by $A_y(x) := B(x, y)$. This operator is continuous by (iii) and hence is bounded by Theorem 1.17. Now fix an element $x \in X$. Then the linear map $Y \to Z : y \mapsto A_y x = B(x, y)$ is continuous by (iii) and hence $\sup_{y \in S} \|A_y x\|_Z < \infty$ by Theorem 1.17. Hence $c := \sup_{y \in S} \|A_y\| < \infty$ by Theorem 2.1. Thus

$$\|B(x, y)\|_Z \leq c \|x\|_X \qquad \text{for all } x \in X \text{ and all } y \in S.$$

This implies (i) and completes the proof of Corollary 2.7.