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1.5 The Baire Category Theorem

The Baire category theorem is a powerful tool in functional analysis. It provides conditions under which a subset of a complete metric space is dense. In fact, it describes a class of dense subsets such that every countable intersection of sets in this class belongs again to this class and hence is still a dense subset. Here are the relevant definitions.

Definition 1.52 (Baire Category). *Let (X, d) be a metric space.*

(i) *A subset $A \subset X$ is called* **nowhere dense** *if the interior of its closure $\overline{A}$ is empty.*

(ii) *A subset $A \subset X$ is said to be* **meagre** *if it is a countable union of nowhere dense subsets of X.*

(iii) *A subset $A \subset X$ is said to be* **nonmeagre** *if it is not meagre.*

(iv) *A subset $A \subset X$ is called* **residual** *if its complement is meagre.*

This definition does not exclude the possibility that X might be the empty set, in which case every subset of X is both meagre and residual. In the literature meagre sets are often called **of the first category**, nonmeagre sets are called **of the second category**, and residual sets are called **comeagre**. The next lemma summarizes some elementary consequences of these definitions.

Lemma 1.53. *Let (X, d) be a metric space. Then the following holds.*

(i) *A subset $A \subset X$ is nowhere dense if and only if its complement $X \setminus A$ contains a dense open subset of X.*

(ii) *If $B \subset X$ is meagre and $A \subset B$ then A is meagre.*

(iii) *If $A \subset X$ is nonmeagre and $A \subset B \subset X$ then B is nonmeagre.*

(iv) *Every countable union of meagre sets is again meagre.*

(v) *Every countable intersection of residual sets is again residual.*

(vi) *A subset of X is residual if and only if it contains a countable intersection of dense open subsets of X.*

Proof. The complement of the closure of a subset of X is the interior of the complement and vice versa. Thus

$$X \setminus \mathrm{int}(\overline{A}) = \overline{X \setminus \overline{A}} = \overline{\mathrm{int}(X \setminus A)}.$$

This shows that a subset $A \subset X$ is nowhere dense if and only if the interior of $X \setminus A$ is dense in X, i.e. $X \setminus A$ contains a dense open subset of X. This proves (i). Parts (ii), (iii), (iv), and (v) follow directly from the definitions.

We prove (vi). Let $R \subset X$ be a residual set and define $A := X \setminus R$. Then there is a sequence of nowhere dense subsets $A_i \subset X$ such that $A = \bigcup_{i=1}^\infty A_i$. Define $U_i := X \setminus \overline{A_i} = \mathrm{int}(X \setminus A_i)$. Then U_i is a dense open set by (i) and

$$\bigcap_{i=1}^\infty U_i = X \setminus \bigcup_{i=1}^\infty \overline{A_i} \subset X \setminus \bigcup_{i=1}^\infty A_i = X \setminus A = R.$$

Conversely, suppose that there is a sequence of dense open subsets $U_i \subset X$ such that $\bigcap_{i=1}^\infty U_i \subset R$. Define $A_i := X \setminus U_i$ and $A := \bigcup_{i=1}^\infty A_i$. Then A_i is nowhere dense by (i) and hence A is meagre by definition. Moreover,

$$X \setminus R \subset X \setminus \bigcap_{i=1}^\infty U_i = \bigcup_{i=1}^\infty (X \setminus U_i) = \bigcup_{i=1}^\infty A_i = A.$$

Hence $X \setminus R$ is meagre by part (ii) and this proves Lemma 1.53. □

Lemma 1.54. *Let (X, d) be a metric space. The following are equivalent.*

(i) *Every residual subset of X is dense.*

(ii) *If $U \subset X$ is a nonempty open set then U is nonmeagre.*

(iii) *If $A_i \subset X$ is a sequence of closed sets with empty interior then their union has empty interior.*

(iv) *If $U_i \subset X$ is a sequence of dense open sets then their intersection is dense in X.*

Proof. We prove that (i) implies (ii). Assume (i) and let $U \subset X$ be a nonempty open set. Then its complement $X \setminus U$ is not dense and so is not residual by (i). Hence U is not meagre.

We prove that (ii) implies (iii). Assume (ii) and let A_i be a sequence of closed subsets of X with empty interior. Then their union A is meagre. Hence the interior of A is also meagre by part (ii) of Lemma 1.53. Hence the interior of A is empty by (ii).

We prove that (iii) implies (iv). Assume (iii) and let U_i be a sequence of dense open subsets of X. Define $A_i := X \setminus U_i$. Then A_i is a sequence of closed subsets of X with empty interior. Hence $A := \bigcup_{i=1}^\infty A_i$ has empty interior by (iii). Hence $R := \bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty (X \setminus A_i) = X \setminus A$ is dense.

We prove that (iv) implies (i). Assume (iv) and let $R \subset X$ be residual. Then, by part (vi) of Lemma 1.53, there exists a sequence of dense open subsets $U_i \subset X$ such that $\bigcap_{i\in\mathbb{N}} U_i \subset R$. By (iv) the set $\bigcap_{i\in\mathbb{N}} U_i$ is dense in X and hence so is R. This proves Lemma 1.54. □

Theorem 1.55 (Baire Category Theorem). *Let (X, d) be a nonempty complete metric space. Then the following holds.*

(i) *Every residual subset of X is dense.*

(ii) *If $U \subset X$ is a nonempty open set then U is nonmeagre.*

(iii) *If $A_i \subset X$ is a sequence of closed sets with empty interior then their union has empty interior.*

(iv) *If $U_i \subset X$ is a sequence of open dense sets then their intersection is dense in X.*

(v) *Every residual subset of X is nonmeagre.*

Proof. The first four assertions are equivalent by Lemma 1.54.

We prove that (ii) implies (v). Let $R \subset X$ be a residual set. Then $X \setminus R$ is meagre by definition. If the set R were meagre as well, then $X = (X \setminus R) \cup R$ would also be meagre by part (iv) of Lemma 1.53, and this would contradict part (ii) because X is nonempty. Thus R is nonmeagre.

We prove part (iv). Thus assume that $U_i \subset X$ is a sequence of dense open sets. Fix an element $x_0 \in X$ and a constant $\varepsilon_0 > 0$. We must prove that $B_{\varepsilon_0}(x_0) \cap \bigcap_{i=1}^{\infty} U_i \neq \emptyset$. We claim that there exist sequences

$$x_k \in U_k, \qquad 0 < \varepsilon_k < 2^{-k}, \qquad k = 1, 2, 3, \ldots, \tag{1.54}$$

such that

$$\overline{B_{\varepsilon_k}(x_k)} \subset U_k \cap B_{\varepsilon_{k-1}}(x_{k-1}) \tag{1.55}$$

for every integer $k \geq 1$. For $k = 1$ observe that $U_1 \cap B_{\varepsilon_0}(x_0)$ is a nonempty open set because U_1 is dense in X. Choose any element $x_1 \in U_1 \cap B_{\varepsilon_0}(x_0)$ and choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < 1/2$ and $\overline{B_{\varepsilon_1}(x_1)} \subset U_1 \cap B_{\varepsilon_0}(x_0)$. Once x_{k-1} and ε_{k-1} have been found for some integer $k \geq 2$, use the fact that U_k is dense in X to find x_k and ε_k such that (1.54) and (1.55) hold.

More precisely, this argument requires the axiom of dependent choice (see page 10). Define the set

$$\mathbf{X} := \left\{ (k, x, \varepsilon) \,\middle|\, k \in \mathbb{N},\, x \in X,\, 0 < \varepsilon < 2^{-k},\, \overline{B_\varepsilon(x)} \subset U_k \cap B_{\varepsilon_0}(x_0) \right\}$$

and define the map $\mathbf{A} : \mathbf{X} \to 2^{\mathbf{X}}$ by

$$\mathbf{A}(k, x, \varepsilon) := \left\{ (k', x', \varepsilon') \in \mathbf{X} \,\middle|\, k' = k + 1,\, \overline{B_{\varepsilon'}(x')} \subset B_\varepsilon(x) \right\}$$

for $(k, x, \varepsilon) \in \mathbf{X}$. Then $\mathbf{X} \neq \emptyset$ and $\mathbf{A}(k, x, \varepsilon) \neq \emptyset$ for all $(k, x, \varepsilon) \in \mathbf{X}$, because U_k is open and dense in X for all k. Hence the existence of the sequences x_k and ε_k follows from the axiom of dependent choice.

Now let $x_k \in U_k$ and $\varepsilon_k > 0$ be sequences that satisfy (1.54) and (1.55). Then $d(x_k, x_{k-1}) < \varepsilon_{k-1} \leq 2^{1-k}$ for all $k \in \mathbb{N}$. Hence

$$d(x_k, x_\ell) \leq \sum_{i=k}^{\ell-1} d(x_i, x_{i+1}) < \sum_{i=k}^{\ell-1} 2^{-i} < 2^{1-k}$$

for all $k, \ell \in \mathbb{N}$ with $\ell > k$. Thus $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete the sequence $(x_k)_{k\in\mathbb{N}}$ converges. Denote its limit by

$$x^* := \lim_{k\to\infty} x_k.$$

Since $x_\ell \in B_{\varepsilon_k}(x_k)$ for every $\ell \geq k$ it follows that

$$x^* \in \overline{B_{\varepsilon_k}(x_k)} \subset U_k \qquad \text{for all } k \in \mathbb{N}.$$

Moreover,

$$x^* \in \overline{B_{\varepsilon_1}(x_1)} \subset B_{\varepsilon_0}(x_0).$$

This shows that the intersection $B_{\varepsilon_0}(x_0) \cap \bigcap_{i=1}^\infty U_i$ is nonempty for all $x_0 \in X$ and all $\varepsilon_0 > 0$. Hence the set $\bigcap_{i=1}^\infty U_i$ is dense in X as claimed. This proves part (iv) and Theorem 1.55. □

The desired class of dense subsets of our nonempty complete metric space is the collection of residual sets. Every residual set is dense by part (i) of Theorem 1.55 and every countable intersection of residual sets is again residual by part (v) of Lemma 1.53. It is often convenient to use the characterization of a residual set as one that contains a countable intersection of dense open sets in part (vi) of Lemma 1.53. A very useful consequence of the Baire Category Theorem is the assertion that a nonempty complete metric space cannot be expressed as a countable union of nowhere dense subsets (part (ii) of Theorem 1.55 with $U = X$).

We emphasize that, while the assumption of the Baire Category Theorem (completeness) depends on the distance function in a crucial way, the conclusion (every countable intersection of dense open subsets is dense) is purely topological. Thus the Baire Category Theorem extends to many metric spaces that are not complete. All that is required is the existence of a complete distance function that induces the same topology as the original distance function.

Example 1.56. Let (M, d) be a complete metric space and let $X \subset M$ be a nonempty open set. Then the conclusions of the Baire Category Theorem hold for the metric space (X, d_X) with $d_X := d|_{X \times X} : X \times X \to [0, \infty)$, even though (X, d_X) may not be complete. To see this, let $U_i \subset X$ be a sequence of dense open subsets of X, choose $x_0 \in X$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x_0) \subset X$, and repeat the argument in the proof of Theorem 1.55 to show that $B_{\varepsilon_0}(x_0) \cap \bigcap_{i=1}^{\infty} U_i \neq \emptyset$. All that is needed is the fact that the closure $\overline{B_{\varepsilon_1}(x_1)}$ that contains the sequence x_k is complete with respect to the induced metric.

Example 1.57. The conclusions of the Baire Category Theorem hold for the topological vector space $X := C^\infty([0, 1])$ of smooth functions $f : [0, 1] \to \mathbb{R}$, equipped with the C^∞ topology. By definition, a sequence $f_n \in C^\infty([0, 1])$ converges to $f \in C^\infty([0, 1])$ with respect to the C^∞ topology if and only if, for each integer $\ell \geq 0$, the sequence of ℓth derivatives $f_n^{(\ell)} : [0, 1] \to \mathbb{R}$ converges uniformly to the ℓth derivative $f^{(\ell)} : [0, 1] \to \mathbb{R}$ as n tends to infinity. This topology is induced by the distance function

$$d(f, g) := \sum_{\ell=0}^{\infty} 2^{-\ell} \frac{\left\| f^{(\ell)} - g^{(\ell)} \right\|_\infty}{1 + \left\| f^{(\ell)} - g^{(\ell)} \right\|_\infty},$$

where $\|u\|_\infty := \sup_{0 \leq t \leq 1} |u(t)|$ denotes the supremum norm of a continuous function $u : [0, 1] \to \mathbb{R}$, and $(C^\infty([0, 1]), d)$ is a complete metric space.

Example 1.58. A residual subset of $\mathbb{R}^n$ may have Lebesgue measure zero. Namely, choose a bijection $\mathbb{N} \to \mathbb{Q}^n : k \mapsto x_k$ and, for $\varepsilon > 0$, define

$$U_\varepsilon := \bigcup_{k=1}^{\infty} B_{2^{-k}\varepsilon}(x_k).$$

This is a dense open subset of $\mathbb{R}^n$ and its Lebesgue measure is less than $(2\varepsilon)^n$. Hence $R := \bigcap_{i=1}^{\infty} U_{1/i}$ is a residual set of Lebesgue measure zero and its complement

$$A := \mathbb{R}^n \setminus R = \bigcup_{i=1}^{\infty} \left(\mathbb{R}^n \setminus U_{1/i} \right)$$

is a meagre set of full Lebesgue measure.

Example 1.59. The conclusions of the Baire category theorem do not hold for the metric space $X = \mathbb{Q}$ of rational numbers with the standard distance function given by $d(x, y) := |x - y|$ for $x, y \in \mathbb{Q}$. Every one element subset of X is nowhere dense and every subset of X is both meagre and residual.