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المادة باللغة الانجليزية	functional analysis
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1.4 Banach Algebras

We begin the discussion with a result about convergent series in a Banach space. It extends the basic assertion in first year analysis that every absolutely convergent series of real numbers converges. We will use Lemma 1.45 to study power series in a Banach algebra.

Lemma 1.45 (Convergent Series). *Let $(X, \|\cdot\|)$ be a Banach space and let $(x_i)_{i\in\mathbb{N}}$ be a sequence in X such that*

$$\sum_{i=1}^{\infty} \|x_i\| < \infty.$$

Then the sequence $\xi_n := \sum_{i=1}^{n} x_i$ in X converges. Its limit is denoted by

$$\sum_{i=1}^{\infty} x_i := \lim_{n\to\infty} \sum_{i=1}^{n} x_i. \tag{1.41}$$

Proof. Define $s_n := \sum_{i=1}^{n} \|x_i\|$ for $n \in \mathbb{N}$. This sequence is nondecreasing and converges by assumption. Moreover, for every pair of integers $n > m \geq 1$, we have $\|\xi_n - \xi_m\| = \|\sum_{i=m+1}^{n} x_i\| \leq \sum_{i=m+1}^{n} \|x_i\| = s_n - s_m$. Hence $(\xi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, this sequence converges, and this proves Lemma 1.45. □

Definition 1.46 (Banach Algebra). *A real (respectively complex)* **Banach algebra** *is a pair consisting of a real (respectively complex) Banach space $(\mathcal{A}, \|\cdot\|)$ and a bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A} : (a, b) \mapsto ab$ (called the* **product***) that is associative, i.e.*

$$(ab)c = a(bc) \qquad \text{for all } a, b, c \in \mathcal{A}, \tag{1.42}$$

and satisfies the inequality

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } a, b \in \mathcal{A}. \tag{1.43}$$

A Banach algebra $\mathcal{A}$ is called **commutative** *if $ab = ba$ for all $a, b \in \mathcal{A}$. It is called* **unital** *if there exists an element $\mathbb{1} \in \mathcal{A} \setminus \{0\}$ such that*

$$\mathbb{1}a = a\mathbb{1} = a \qquad \text{for all } a \in \mathcal{A}. \tag{1.44}$$

The **unit** *$\mathbb{1}$, if it exists, is uniquely determined by the product. An element $a \in \mathcal{A}$ of a unital Banach algebra $\mathcal{A}$ is called* **invertible** *if there exists an element $b \in \mathcal{A}$ such that $ab = ba = \mathbb{1}$. The element b, if it exists, is uniquely determined by a, is called the* **inverse of** *a, and is denoted by $a^{-1} := b$. The invertible elements form a group $\mathcal{G} \subset \mathcal{A}$.*

Example 1.47. **(i)** The archetypal example of a Banach algebra is the space $\mathcal{L}(X) := \mathcal{L}(X, X)$ of bounded linear operators from a Banach space X to itself with the operator norm (Definition 1.16 and Theorem 1.31). This Banach algebra is unital whenever $X \neq \{0\}$ and the unit is the identity. It turns out that the invertible elements of $\mathcal{L}(X)$ are the bijective bounded linear operators from X to itself. That the inverse of a bijective bounded linear operator is again a bounded linear operator is a nontrivial result. It follows from the Open Mapping Theorem proved in Section 2.2 below.

(ii) An example of a commutative unital Banach algebra is the space of real valued bounded continuous functions on a nonempty topological space equipped with the supremum norm and pointwise multiplication.

(iii) A third example of a unital Banach algebra is the space $\ell^1(\mathbb{Z})$ of bi-infinite summable sequences $(x_i)_{i\in\mathbb{Z}}$ of real numbers with the convolution product defined by $(x * y)_i := \sum_{j\in\mathbb{Z}} x_j y_{i-j}$ for $x, y \in \ell^1(\mathbb{Z})$.

(iv) A fourth example of a Banach algebra is the space $L^1(\mathbb{R}^n)$ of Lebesgue integrable functions on $\mathbb{R}^n$ (modulo equality almost everywhere), where multiplication is given by convolution (see [50, Section 7.5]). This Banach algebra does not admit a unit. A candidate for a unit would be the Dirac delta function at the origin which is not actually a function but a measure. The convolution product extends to the space of signed Borel measures on $\mathbb{R}^n$ and they form a commutative unital Banach algebra.

Let $\mathcal{A}$ be a complex Banach algebra and let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1.45}$$

be a power series with complex coefficients $c_n \in \mathbb{C}$ and convergence radius

$$\rho := \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}} > 0. \tag{1.46}$$

Choose an element $a \in \mathcal{A}$ with $\|a\| < \rho$. Then the sequence $(c_n a^n)_{n\in\mathbb{N}}$ satisfies the inequality $\sum_{n=0}^{\infty} \|c_n a^n\| \le |c_0| \, \|\mathbb{1}\| + \sum_{n=1}^{\infty} |c_n| \, \|a\|^n < \infty$ and so the sequence $\xi_n := \sum_{i=0}^{n} c_i a^i$ converges by Lemma 1.45. Denote the limit by

$$f(a) := \sum_{n=0}^{\infty} c_n a^n \tag{1.47}$$

for $a \in \mathcal{A}$ with $\|a\| < \rho$. When the power series f has real coefficients, this definition extends to real Banach algebras.

Exercise 1.48. The map $f : \{a \in \mathcal{A} \mid \|a\| < \rho\} \to \mathcal{A}$ defined by (1.47) is continuous. **Hint:** For $n \in \mathbb{N}$ define $f_n : X \to X$ by $f_n(a) := \sum_{i=0}^{n} c_i a^i$. Prove that f_n is continuous. Prove that the sequence f_n converges uniformly to f on the set $\{a \in \mathcal{A} \mid \|a\| \le r\}$ for every $r < \rho$.

Theorem 1.49 (Inverse). *Let $\mathcal{A}$ be a real unital Banach algebra.*

(i) *For every $a \in \mathcal{A}$ the limit*

$$r_a := \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \le \|a\| \tag{1.48}$$

exists. It is called the **spectral radius** *of a.*

(ii) *If $a \in \mathcal{A}$ satisfies $r_a < 1$ then the element $\mathbb{1} - a$ is invertible and*

$$(\mathbb{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n. \tag{1.49}$$

(iii) *The group $\mathcal{G} \subset \mathcal{A}$ of invertible elements is an open subset of $\mathcal{A}$ and the map $\mathcal{G} \to \mathcal{G} : a \mapsto a^{-1}$ is continuous. More precisely, if $a \in \mathcal{G}$ and $b \in \mathcal{A}$ satisfy $\|a - b\|\|a^{-1}\| < 1$, then $b \in \mathcal{G}$ and $b^{-1} = \sum_{n=0}^{\infty}(\mathbb{1} - a^{-1}b)^n a^{-1}$ and*

$$\|b^{-1} - a^{-1}\| \le \frac{\|a - b\|\|a^{-1}\|^2}{1 - \|a - b\|\|a^{-1}\|}, \qquad \|b^{-1}\| \le \frac{\|a^{-1}\|}{1 - \|a - b\|\|a^{-1}\|}. \tag{1.50}$$

Proof. We prove part (i). Let $a \in \mathcal{A}$, define $r := \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \ge 0$, and fix a real number $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\|a^m\|^{1/m} < r + \varepsilon$ and define

$$M := \max_{\ell=0,1,\dots,m-1} \left(\frac{\|a\|}{r+\varepsilon}\right)^{\ell}.$$

Fix two integers $k \ge 0$ and $0 \le \ell \le m-1$ and let $n := km + \ell$. Then

$$\begin{aligned}
\|a^n\|^{1/n} &= \left\|a^{km}a^{\ell}\right\|^{1/n} \\
&\le \|a\|^{\ell/n} \|a^m\|^{k/n} \\
&\le \|a\|^{\ell/n} (r+\varepsilon)^{km/n} \\
&= \left(\frac{\|a\|}{r+\varepsilon}\right)^{\ell/n} (r+\varepsilon) \\
&\le M^{1/n}(r+\varepsilon).
\end{aligned}$$

Since $\lim_{n\to\infty} M^{1/n} = 1$, there is an integer $n_0 \in \mathbb{N}$ such that $\|a^n\|^{1/n} < r + 2\varepsilon$ for every integer $n \ge n_0$. Hence the limit r_a in (1.48) exists and is equal to r. This proves part (i).

We prove part (ii). Let $a \in \mathcal{A}$ and assume $r_a < 1$. Choose a real number α such that $r_a < \alpha < 1$. Then there exists an $n_0 \in \mathbb{N}$ such that $\|a^n\|^{1/n} \leq \alpha$ for every integer $n \geq n_0$. Hence

$$\|a^n\| \leq \alpha^n \qquad \text{for every integer } n \geq n_0.$$

This implies $\sum_{n=0}^{\infty} \|a^n\| < \infty$, so the sequence

$$b_n := \sum_{i=0}^{n} a^i$$

converges by Lemma 1.45. Denote the limit by b. Since

$$b_n(\mathbb{1} - a) = (\mathbb{1} - a)b_n = \mathbb{1} - a^{n+1}$$

for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|a^{n+1}\| \leq \lim_{n\to\infty} \alpha^{n+1} = 0$, it follows that

$$b(\mathbb{1} - a) = (\mathbb{1} - a)b = \mathbb{1}.$$

Hence $\mathbb{1} - a$ is invertible and $(\mathbb{1} - a)^{-1} = b$. This proves part (ii).

We prove part (iii). Fix an element $a \in \mathcal{G}$ and let $b \in \mathcal{A}$ such that

$$\|a - b\| \, \|a^{-1}\| < 1.$$

Then $\|\mathbb{1} - a^{-1}b\| < 1$ and hence

$$a^{-1}b = \mathbb{1} - (\mathbb{1} - a^{-1}b) \in \mathcal{G}, \qquad (a^{-1}b)^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - a^{-1}b)^n$$

by part (ii). Hence $b = a(a^{-1}b) \in \mathcal{G}$ and

$$b^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - a^{-1}b)^n a^{-1}$$

and so

$$\begin{aligned} \|b^{-1} - a^{-1}\| &\leq \sum_{n=1}^{\infty} \|a - b\|^n \|a^{-1}\|^{n+1} \\ &= \frac{\|a - b\| \|a^{-1}\|^2}{1 - \|a - b\| \|a^{-1}\|}. \end{aligned}$$

Thus $B_{\|a^{-1}\|^{-1}}(a) \subset \mathcal{G}$ and the map $B_{\|a^{-1}\|^{-1}}(a) \to \mathcal{G} : b \mapsto b^{-1}$ is continuous. This proves part (iii) and Theorem 1.49. $\square$

Definition 1.50 (Invertible Operator). *Let X and Y be Banach spaces. A bounded linear operator $A : X \to Y$ is called* **invertible**, *if there exists a bounded linear operator $B : Y \to X$ such that*

$$BA = \mathbb{1}_X, \qquad AB = \mathbb{1}_Y.$$

The operator B is uniquely determined by A and is denoted by

$$B =: A^{-1}.$$

It is called the **inverse** *of A. When $X = Y$, the space of invertible bounded linear operators in $\mathcal{L}(X)$ is denoted by*

$$\mathrm{Aut}(X) := \left\{ A \in \mathcal{L}(X) \,\middle|\, \text{there is a } B \in \mathcal{L}(X) \text{ such that } AB = BA = \mathbb{1} \right\}.$$

The **spectral radius** *of a bounded linear operator $A \in \mathcal{L}(X)$ is the real number $r_A \geq 0$ defined by*

$$r_A := \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|^{1/n} \leq \|A\|. \qquad (1.51)$$

Corollary 1.51 (Spectral Radius). *Let X and Y be Banach spaces. Then the following holds.*

(i) *If $A \in \mathcal{L}(X)$ has spectral radius $r_A < 1$ then*

$$\mathbb{1}_X - A \in \mathrm{Aut}(X), \qquad (\mathbb{1}_X - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

(ii) *$\mathrm{Aut}(X)$ is an open subset of $\mathcal{L}(X)$ with respect to the norm topology and the map $\mathrm{Aut}(X) \to \mathrm{Aut}(X) : A \mapsto A^{-1}$ is continuous.*

(iii) *Let $A, P \in \mathcal{L}(X, Y)$ be bounded linear operators. Assume A is invertible and $\|P\|\|A^{-1}\| < 1$. Then $A - P$ is invertible,*

$$(A - P)^{-1} = \sum_{n=0}^{\infty} (A^{-1}P)^n A^{-1}, \qquad (1.52)$$

and

$$\|(A - P)^{-1} - A^{-1}\| \leq \frac{\|P\|\|A^{-1}\|^2}{1 - \|P\|\|A^{-1}\|}. \qquad (1.53)$$

Proof. Assertions (i) and (ii) follow from Theorem 1.49 with $\mathcal{A} = \mathcal{L}(X)$. To prove part (iii), observe that $\|A^{-1}P\| \leq \|A^{-1}\|\|P\| < 1$. Hence it follows from part (i) that the operator $\mathbb{1}_X - A^{-1}P$ is invertible and that its inverse is given by $(\mathbb{1}_X - A^{-1}P)^{-1} = \sum_{k=0}^{\infty} (A^{-1}P)^k$. Multiply this identity by A^{-1} on the right to obtain (1.52). The inequality (1.53) follows directly from (1.52) and the limit formula for a geometric series. This proves Corollary 1.51. □