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عنوان المحاضرة باللغة الانجليزية	The Dual Space
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1.3 The Dual Space

1.3.1 The Banach Space of Bounded Linear Operators

This section returns to the normed vector space $\mathcal{L}(X, Y)$ of bounded linear operators from X to Y introduced in Definition 1.16. The next theorem shows that $\mathcal{L}(X, Y)$ is complete whenever the target space Y is complete, even if X is not complete.

Theorem 1.31. *Let X be a normed vector space and let Y be a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space with respect to the operator norm.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. Then

$$\|A_n x - A_m x\|_Y = \|(A_n - A_m)x\|_Y \le \|A_n - A_m\| \, \|x\|_X$$

for all $x \in X$ and all $m, n \in \mathbb{N}$. Hence $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y for every $x \in X$. Since Y is complete, this implies that the limit

$$Ax := \lim_{n\to\infty} A_n x \tag{1.20}$$

exists for all $x \in X$. This defines a map $A : X \to Y$. That it is linear follows from the definition, the fact that the limit of a sum of two sequences is the sum of the limits, and the fact that the limit of a product of a sequence with a scalar is the product of the limit with the scalar.

It remains to prove that A is bounded and that $\lim_{n\to\infty} \|A - A_n\| = 0$. To see this, fix a constant $\varepsilon > 0$. Since $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the operator norm, there exists an integer $n_0 \in \mathbb{N}$ such that

$$m, n \in \mathbb{N}, \qquad m, n \ge n_0 \qquad \Longrightarrow \qquad \|A_m - A_n\| < \varepsilon.$$

This implies

$$\begin{aligned} \|Ax - A_n x\|_Y &= \lim_{m\to\infty} \|A_m x - A_n x\|_Y \\ &\le \limsup_{m\to\infty} \|A_m - A_n\| \, \|x\|_X \\ &\le \varepsilon \|x\|_X \end{aligned} \tag{1.21}$$

for every $x \in X$ and every integer $n \ge n_0$. Hence

$$\|Ax\|_Y \le \|Ax - A_{n_0} x\|_Y + \|A_{n_0} x\|_Y \le (\varepsilon + \|A_{n_0}\|) \, \|x\|_X$$

for all $x \in X$ and so A is bounded. It follows also from (1.21) that, for every $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ such that $\|A - A_n\| \le \varepsilon$ for every integer $n \ge n_0$. Thus $\lim_{n\to\infty} \|A - A_n\| = 0$ and this proves Theorem 1.31. □

1.3.2 Examples of Dual Spaces

An important special case is where the target space Y is the real axis. Then Theorem 1.31 asserts that the space

$$X^* := \mathcal{L}(X, \mathbb{R}) \tag{1.22}$$

of bounded linear functionals $\Lambda : X \to \mathbb{R}$ is a Banach space for every normed vector space X (whether or not X is itself complete). The space of bounded linear functionals on X is called the **dual space of** X. The dual space of a Banach space plays a central role in functional analysis. Here are several examples of dual spaces.

Example 1.32 (Dual Space of a Hilbert Space). Let H be a Hilbert space, i.e. H is a Banach space and the norm on H arises from an inner product $H \times H \to \mathbb{R} : (x, y) \mapsto \langle x, y \rangle$ via $\|x\| = \sqrt{\langle x, x \rangle}$. Then every element $y \in H$ determines a linear functional $\Lambda_y : H \to \mathbb{R}$ defined by

$$\Lambda_y(x) := \langle x, y \rangle \qquad \text{for } x \in H. \tag{1.23}$$

It is bounded by the Cauchy–Schwarz inequality (Lemma 1.40) and the Riesz Representation Theorem asserts that the map $H \to H^* : y \mapsto \Lambda_y$ is an isometric isomorphism (Theorem 1.43).

Example 1.33 (Dual Space of $L^p(\mu)$). Let $(M, \mathcal{A}, \mu)$ be a measure space and fix a constant $1 < p < \infty$. Define the number $1 < q < \infty$ by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.24}$$

Then the **Hölder inequality** asserts that the product of two functions $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$ is μ-integrable and satisfies the inequality

$$\left| \int_M fg \, d\mu \right| \le \|f\|_p \, \|g\|_q \,. \tag{1.25}$$

(See [50, Theorem 4.1].) This implies that every $g \in L^q(\mu)$ determines a bounded linear functional $\Lambda_g : L^p(\mu) \to \mathbb{R}$ defined by

$$\Lambda_g(f) := \int_M fg \, d\mu \qquad \text{for } f \in L^p(\mu). \tag{1.26}$$

It turns out that

$$\|\Lambda_g\|_{\mathcal{L}(L^p(\mu),\mathbb{R})} = \|g\|_q$$

for all $g \in L^q(\mu)$ (see [50, Theorem 4.33]) and that the map

$$L^q(\mu) \to L^p(\mu)^* : g \mapsto \Lambda_g$$

is an isometric isomorphism (see [50, Thm 4.35]). The proof relies on the Radon–Nikodým Theorem (see [50, Thm 5.4]).

This result extends to the case $p = 1$ and shows that the natural map

$$L^\infty(\mu) \to L^1(\mu)^* : g \mapsto \Lambda_g$$

is an isometric isomorphism if and only if the measure space $(M, \mathcal{A}, \mu)$ is localizable. In particular, the dual space of $L^1(\mu)$ is isomorphic to $L^\infty(\mu)$ whenever $(M, \mathcal{A}, \mu)$ is a σ-finite measure space. (See [50, Def 4.29] for the relevant definitions.) However, the dual space of $L^\infty(\mu)$ is in general much larger than $L^1(\mu)$, i.e. the map

$$L^1(\mu) \to L^\infty(\mu)^* : g \mapsto \Lambda_g$$

in (1.26) is an isometric embedding but is typically far from surjective.

Example 1.34 (Dual Space of ℓ^p). Fix a number $1 < p < \infty$ and consider the Banach space ℓ^p of p-summable sequences of real numbers, equipped with the norm

$$\|x\|_p := \left(\sum_{i=1}^{p} |x_i|^p \right)^{1/p} \qquad \text{for } x = (x_i)_{i \in \mathbb{N}} \in \ell^p.$$

(See part (ii) of Example 1.3.) This is the special case of the counting measure on $M = \mathbb{N}$ in Example 1.33 and so the dual space of ℓ^p is isomorphic to ℓ^q, where $1/p + 1/q = 1$. Here is a proof in this special case.

Associated to every sequence $y = (y_i)_{i \in \mathbb{N}} \in \ell^q$ is a bounded linear functional $\Lambda_y : \ell^p \to \mathbb{R}$, defined by

$$\Lambda_y(x) := \sum_{i=1}^{\infty} x_i y_i \tag{1.27}$$

for $x = (x_i)_{i \in \mathbb{N}} \in \ell^p$. It is well defined by the Hölder inequality (1.25). Namely, in this case the Hölder inequality takes the form

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q$$

for $x = (x_i)_{i\in\mathbb{N}} \in \ell^p$ and $y = (y_i)_{i\in\mathbb{N}} \in \ell^q$ and hence the limit

$$\sum_{i=1}^{\infty} x_i y_i = \lim_{n\to\infty} \sum_{i=1}^{n} x_i y_i$$

in (1.27) exists. Thus, for each $y \in \ell^q$, the map

$$\Lambda_y : \ell^p \to \mathbb{R}$$

in (1.27) is well defined and linear and satisfies the inequality

$$|\Lambda_y(x)| \le \|x\|_p \|y\|_q$$

for all $x \in \ell^p$. Thus Λ_y is a bounded linear functional on ℓ^p for every $y \in \ell^q$ with norm

$$\|\Lambda_y\| = \sup_{x\in\ell^p\setminus\{0\}} \frac{|\Lambda_y(x)|}{\|x\|_p} \le \|y\|_q .$$

Hence the formula (1.27) defines a bounded linear operator

$$\ell^q \to (\ell^p)^* : y \mapsto \Lambda_y. \tag{1.28}$$

In fact, it turns out that $\|\Lambda_y\| = \|y\|_q$ for all $y \in \ell^q$. To see this, fix a nonzero element $y = (y_i)_{i\in\mathbb{N}} \in \ell^q$ and consider the sequence $x = (x_i)_{i\in\mathbb{N}}$, defined by $x_i := |y_i|^{q-1}\mathrm{sign}(y_i)$ for $i \in \mathbb{N}$, where $\mathrm{sign}(y_i) := 1$ when $y_i \ge 0$ and $\mathrm{sign}(y_i) := -1$ when $y_i < 0$. Then $|x_i|^p = |y_i|^{(q-1)p} = |y_i|^q$ and thus

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |y_i|^q\right)^{1-1/q} = \|y\|_q^{q-1}, \qquad \Lambda_y(x) = \sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} |y_i|^q = \|y\|_q^q .$$

This shows that

$$\|\Lambda_y\| \ge \frac{|\Lambda_y(x)|}{\|x\|_p} = \frac{\|y\|_q^q}{\|y\|_q^{q-1}} = \|y\|_q$$

and so $\|\Lambda_y\| = \|y\|_q$. Thus the map (1.28) is an isometric embedding.

We prove that it is surjective. For $i \in \mathbb{N}$ define

$$e_i := (\delta_{ij})_{j\in\mathbb{N}} \tag{1.29}$$

where δ_{ij} denotes the **Kronecker symbol**, i.e. $\delta_{ij} := 1$ for $i = j$ and $\delta_{ij} := 0$ for $i \neq j$. Then $e_i \in \ell^p$ for every $i \in \mathbb{N}$ and the subspace $\mathrm{span}\{e_i \,|\, i \in \mathbb{N}\}$ of all (finite) linear combinations of the e_i is dense in ℓ^p. Let $\Lambda : \ell^p \to \mathbb{R}$ be a nonzero bounded linear functional and define $y_i := \Lambda(e_i)$ for $i \in \mathbb{N}$. Since $\Lambda \neq 0$ there is an $i \in \mathbb{N}$ such that $y_i \neq 0$. Consider the sequences

$$\xi_n := \sum_{i=1}^{n} |y_i|^{q-1} \mathrm{sign}(y_i) e_i \in \ell^p, \qquad \eta_n := \sum_{i=1}^{n} y_i e_i \in \ell^q \qquad \text{for } n \in \mathbb{N}.$$

Since $(q-1)p = q$, they satisfy

$$\|\xi_n\|_p = \left(\sum_{i=1}^{n} |y_i|^q\right)^{1-1/q} = \|\eta_n\|_q^{q-1}, \qquad \Lambda(\xi_n) = \sum_{i=1}^{n} |y_i|^q = \|\eta_n\|_q^q,$$

and so

$$\left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} = \|\eta_n\|_q = \frac{\Lambda(\xi_n)}{\|\xi_n\|_p} \leq \|\Lambda\|$$

for $n \in \mathbb{N}$ sufficiently large. Thus $y = (y_i)_{i \in \mathbb{N}} \in \ell^q$. Since $\Lambda_y(e_i) = \Lambda(e_i)$ for all $i \in \mathbb{N}$ and the linear subspace $\mathrm{span}\{e_i \,|\, i \in \mathbb{N}\}$ is dense in ℓ^p, it follows that $\Lambda_y = \Lambda$. This proves that the map (1.27) is an isometric isomorphism.

Example 1.35 (Dual Space of ℓ^1). The discussion of Example 1.34 extends to the case $p = 1$ and shows that the natural map

$$\ell^\infty \to (\ell^1)^* : y \mapsto \Lambda_y$$

defined by (1.27) is a Banach space isometry. Here $\ell^\infty \subset \mathbb{R}^{\mathbb{N}}$ is the space of bounded sequences of real numbers equipped with the supremum norm. (**Exercise:** Prove this by adapting Example 1.34 to the case $p = 1$.)

There is an analogous map $\ell^1 \to (\ell^\infty)^* : y \mapsto \Lambda_y$. This map is again an isometric embedding of Banach spaces, however, it is far from surjective. The existence of a linear functional on ℓ^∞ that cannot be represented by a summable sequence can be established via the Hahn–Banach Theorem.

Example 1.36 (Dual Space of c_0). Consider the closed linear subspace of ℓ^∞ which consists of all sequences of real numbers that converge to zero. Denote it by

$$c_0 := \left\{ x = (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \lim_{i \to \infty} x_i = 0 \right\} \subset \ell^\infty. \tag{1.30}$$

This is a Banach space with the supremum norm $\|x\|_\infty := \sup_{i\in\mathbb{N}}|x_i|$. Every summable sequence $y = (y_i)_{i\in\mathbb{N}} \in \ell^1$ defines a linear functional $\Lambda_y : c_0 \to \mathbb{R}$ via (1.27). It is bounded and $\|\Lambda_y\| \le \|y\|_1$ because

$$|\Lambda_y(x)| \le \sum_{i=1}^{\infty}|x_i y_i| \le \|x\|_\infty \sum_{i=1}^{\infty}|y_i| = \|x\|_\infty \|y\|_1$$

for all $x \in c_0$. Thus the map

$$\ell^1 \to c_0^* : y \mapsto \Lambda_y \tag{1.31}$$

is a bounded linear operator. In fact, it is an isometric isomorphism of Banach spaces. To see this, fix an element $y = (y_i)_{i\in\mathbb{N}} \in \ell^1$ and define $\varepsilon_i := \operatorname{sign}(y_i)$ for $i \in \mathbb{N}$. Thus $\varepsilon_i = 1$ when $y_i \ge 0$ and $\varepsilon_i = -1$ when $y_i < 0$. For $n \in \mathbb{N}$ define $\xi_n := \sum_{i=1}^{n} \varepsilon_i e_i \in c_0$, where $e_i \in c_0$ is defined by (1.29). Then $\Lambda_y(\xi_n) = \sum_{i=1}^{n}|y_i|$ and $\|\xi_n\|_\infty = 1$. Thus $\|\Lambda_y\| \ge \sum_{i=1}^{n}|y_i|$ for all $n \in \mathbb{N}$, hence

$$\|\Lambda_y\| \ge \sum_{i=1}^{\infty}|y_i| = \|y\|_1 \ge \|\Lambda_y\|,$$

and so $\|\Lambda_y\| = \|y\|_1$. This shows that the linear map (1.31) is an isometric embedding and, in particular, is injective.

We prove that the map (1.31) is surjective. Let $\Lambda : c_0 \to \mathbb{R}$ be a nonzero bounded linear functional and define the sequence $y = (y_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ by $y_i := \Lambda(e_i)$ for $i \in \mathbb{N}$ where $e_i \in c_0$ is the sequence in (1.29). As before, define $\xi_n := \sum_{i=1}^{n} \operatorname{sign}(y_i) e_i \in c_0$ for $n \in \mathbb{N}$. Then $\|\xi_n\| = 1$ for n sufficiently large and therefore

$$\sum_{i=1}^{n}|y_i| = \Lambda(\xi_n) \le \|\Lambda\| \qquad \text{for all } n \in \mathbb{N}.$$

This implies $\|y\|_1 = \sum_{i=1}^{\infty}|y_i| \le \|\Lambda\|$ and so $y \in \ell^1$. Since $\Lambda_y(e_i) = y_i = \Lambda(e_i)$ for all $i \in \mathbb{N}$ and the linear subspace $\operatorname{span}\{e_i \,|\, i \in \mathbb{N}\}$ is dense in c_0 (prove this!), it follows that $\Lambda_y = \Lambda$. Hence the map (1.31) is a Banach space isometry and so $c_0^* \cong \ell^1$.

Example 1.37 (Dual Space of $C(M)$). Let M be a second countable compact Hausdorff space, so M is metrizable [40]. Denote by $\mathcal{B} \subset 2^M$ its **Borel σ-algebra**, i.e. the smallest σ-algebra containing the open sets. Consider the Banach space $C(M)$ of continuous real valued functions on M with the supremum norm and denote by $\mathcal{M}(M)$ the Banach space of signed Borel measures $\mu : \mathcal{B} \to \mathbb{R}$ with the norm in equation (1.4) (see Example 1.3). Every signed Borel measure $\mu : \mathcal{B} \to \mathbb{R}$ determines a bounded linear functional $\Lambda_\mu : C(M) \to \mathbb{R}$ defined by

$$\Lambda_\mu(f) := \int_M f\, d\mu \qquad \text{for } f \in C(M). \tag{1.32}$$

The Hahn Decomposition Theorem asserts that for every signed Borel measure $\mu : \mathcal{B} \to \mathbb{R}$ there exists a Borel set $P \subset M$ such that $\mu(B \cap P) \geq 0$ and $\mu(B \setminus P) \leq 0$ for every Borel set $B \subset M$ (see [50, Thm 5.19]). Since every Borel measure on M is regular (see [50, Def 3.1 and Thm 3.18]) this can be used to show that $\|\Lambda_\mu\|_{\mathcal{L}(C(M),\mathbb{R})} = \|\mu\|$. Now every bounded linear functional $\Lambda : C(M) \to \mathbb{R}$ can be expressed as the difference of two positive linear functionals $\Lambda^\pm : C(M) \to \mathbb{R}$ (see [50, Ex 5.35]). Hence it follows from the Riesz Representation Theorem (see [50, Cor 3.19]) that the linear map $\mathcal{M}(M) \to C(M)^* : \mu \mapsto \Lambda_\mu$ is an isometric isomorphism.

Exercise 1.38. Let X be an infinite-dimensional normed vector space and let $\Lambda : X \to \mathbb{R}$ be a nonzero linear functional. The following are equivalent.

(i) Λ is bounded.

(ii) The kernel of Λ is a closed linear subspace of X.

(iii) The kernel of Λ is not dense in X.