الكلية	**التربية للعلوم الصرفة**
القسم	**الرياضيات**
المادة باللغة الانجليزية	functional analysis
المادة باللغة العربية	**التحليل الدالي**
المرحلة الدراسية	**المرحلة الرابعة**
اسم التدريسي	**احمد محمد مصطفى**
عنوان المحاضرة باللغة الانجليزية	Quotient and Product Spaces
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محتوى المحاضرة

1.2.3 Quotient and Product Spaces

Quotient Spaces

Let $(X, \|\cdot\|)$ be a real normed vector space and let $Y \subset X$ be a closed subspace. Define an equivalence relation $\sim$ on X by

$$x \sim x' \qquad \Longleftrightarrow \qquad x' - x \in Y.$$

Denote the equivalence class of an element $x \in X$ under this equivalence relation by $[x] := x + Y := \left\{x + y \,\middle|\, y \in Y\right\}$ and denote the quotient space by

$$X/Y := \left\{x + Y \,\middle|\, x \in X\right\}.$$

For $x \in X$ define

$$\|[x]\|_{X/Y} := \inf_{y \in Y} \|x + y\|_X . \tag{1.17}$$

Then X/Y is a real vector space and the formula (1.17) defines a norm function on X/Y. (**Exercise:** Prove this.) The next lemma is the key step in the proof that if X is a Banach space so the quotient space X/Y for every closed linear subspace $Y \subset X$.

Lemma 1.28. *Let X be a normed vector space and let $Y \subset X$ be a closed linear subspace. let $(x_i)_{i \in \mathbb{N}}$ be a sequence in X such that $([x_i])_{i \in \mathbb{N}}$ is a Cauchy sequence in X/Y with respect to the norm 1.17. Then there exists a subsequence $(x_{i_k})_{k \in \mathbb{N}}$ and a sequence $(y_k)_{k \in \mathbb{N}}$ in Y such that $(x_{i_k} + y_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in X.*

Proof. Choose $i_1 := 1$ and let $i_2 > i_1$ be the smallest integer bigger than i_1 such that $\inf_{y \in Y} \|x_{i_1} - x_{i_2} + y\|_X < 2^{-1}$. Once $i_1, \dots, i_k$ have been constructed, choose $i_{k+1} > i_k$ to be the smallest integer bigger than i_k such that $\inf_{y \in Y} \left\|x_{i_k} - x_{i_{k+1}} + y\right\|_X < 2^{-k}$. This completes the inductive construction of the subsequence $(x_{i_k})_{k \in \mathbb{N}}$. Now use the Axiom of Countable Choice to find a sequence $(\eta_k)_{k \in \mathbb{N}}$ in Y such that $\left\|x_{i_k} - x_{i_{k+1}} + \eta_k\right\|_X < 2^{-k}$ for all $k \in \mathbb{N}$. Define

$$y_1 := 0, \qquad y_k := -\eta_1 - \cdots - \eta_{k-1} \qquad \text{for } k \geq 2.$$

Then

$$\left\|x_{i_k} + y_k - x_{i_{k+1}} - y_{k+1}\right\|_X = \left\|x_{i_k} - x_{i_{k+1}} + \eta_k\right\|_X < 2^{-k}$$

for all $k \in \mathbb{N}$ and hence $(x_{i_k} + y_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. This proves Lemma 1.28. □

Theorem 1.29 (Quotient Space). *Let X be a normed vector space and let $Y \subset X$ be a closed linear subspace. Then the following holds.*

(i) *The map $\pi : X \to X/Y$ defined by $\pi(x) := [x] = x + Y$ for $x \in X$ is a surjective bounded linear operator.*

(ii) *Let $A : X \to Z$ be a bounded linear operator with values in a normed vector space Z such that $Y \subset \ker(A)$. Then there exists a unique bounded linear operator $A_0 : X/Y \to Z$ such that $A_0 \circ \pi = A$.*

(iii) *If X is a Banach space then X/Y is a Banach space.*

Proof. Part (i) follows directly from the definitions.

To prove part (ii) observe that the operator $A_0 : X/Y \to Z$, given by

$$A_0[x] := Ax \qquad \text{for } x \in X,$$

is well defined whenever $Y \subset \ker(A)$. It is obviously linear and it satisfies

$$\|A_0[x]\|_Z = \|A(x+y)\|_Z \leq \|A\| \, \|x+y\|_X$$

for all $x \in X$ and all $y \in Y$. Take the infimum over all $y \in Y$ to obtain the inequality

$$\|A_0[x]\|_Z \leq \inf_{y \in Y} \|A\| \, \|x+y\|_X = \|A\| \, \|[x]\|_{X/Y}$$

for all $x \in X$. This proves part (ii).

To prove part (iii), assume X is complete and let $(x_i)_{i \in \mathbb{N}}$ be a sequence in X such that $([x_i])_{i \in \mathbb{N}}$ is a Cauchy sequence in X/Y with respect to the norm (1.17). By Lemma 1.28 there exists a subsequence $(x_{i_k})_{k \in \mathbb{N}}$ and a sequence $(y_k)_{k \in \mathbb{N}}$ in Y such that $(x_{i_k} + y_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in X. Since X is a Banach space, there exists an element $x \in X$ such that

$$\lim_{k \to \infty} \|x - x_{i_k} - y_k\|_X = 0.$$

Hence

$$\lim_{k \to \infty} \|[x - x_{i_k}]\|_{X/Y} = \lim_{k \to \infty} \inf_{y \in Y} \|x - x_{i_k} + y\|_X = 0.$$

Thus the subsequence $([x_{i_k}])_{k \in \mathbb{N}}$ converges to $[x]$ in X/Y. Since a Cauchy sequence converges whenever it has a convergent subsequence, this proves Theorem 1.29. □

Product Spaces

Let X and Y be normed vector spaces. Then the product space $X \times Y$ admits the structure of a normed vector space. However, there is no canoncial norm on this product space although it has a canonical product topology (page 118). Examples of norms that induce the product topology are

$$\|(x, y)\|_p := \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p}, \qquad 1 \leq p < \infty, \tag{1.18}$$

and

$$\|(x, y)\|_\infty := \max\left\{\|x\|_X, \|y\|_Y\right\} \tag{1.19}$$

for $x \in X$ and $y \in Y$.

Exercise 1.30. **(i)** Show that the norms in (1.18) and (1.19) are all equivalent and induce the product topology on $X \times Y$.

(ii) Show that the product space $X \times Y$, with any of the norms in (1.18) or (1.19), is a Banach space if and only if X and Y are Banach spaces.