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فضاءات بناخ منتهية البعد

1.2 Finite-Dimensional Banach Spaces

The purpose of the present section is to examine finite-dimensional normed vector spaces with an emphasis on those properties that distinguish them from infinite-dimensional normed vector spaces, which are the main subject of functional analysis. Finite-dimensional normed vector spaces are complete, their linear subspaces are closed, linear functionals on them are continuous, and their closed unit balls are compact. Theorem 1.26 below shows that this last property characterizes finite-dimensionality. Before entering into the main topic of this section, it is convenient to first introduce the concept of a bounded linear operator.

1.2.1 Bounded Linear Operators

The second fundamental concept in functional analysis, after that of a Banach space, is the notion of a bounded linear operator. In functional analysis it is common practice to use the term *linear operator* instead of *linear map*, although both terms have the exact same meaning, namely that of a map between vector spaces that preserves addition and scalar multiplication. The reason lies in the fact that the relevant normed vector spaces in applications are often function spaces and then the elements of the space on which the operator acts are themselves functions. If domain and target of a linear operator are *normed* vector spaces, it is natural to impose continuity with respect to the norm topologies. This underlies the following definition.

Definition 1.16 (Bounded Linear Operator).
Let $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be real normed vector spaces. A linear operator* $A : X \to Y$ *is called* **bounded** *if there exists a constant* $c \geq 0$ *such that*

$$\|Ax\|_Y \leq c\,\|x\|_X \qquad \text{for all } x \in X. \tag{1.14}$$

The smallest constant $c \geq 0$ *that satisfies* (1.14) *is called the* **operator norm of** A *and is denoted by*

$$\|A\| := \|A\|_{\mathcal{L}(X,Y)} := \sup_{x \in X \setminus \{0\}} \frac{\|Ax\|_Y}{\|x\|_X}. \tag{1.15}$$

A bounded linear operator with values in $Y = \mathbb{R}$ *is called a* **bounded linear functional** *on* X*. The space of bounded linear operators from* X *to* Y *is* *denoted by*[1]

$$\mathcal{L}(X,Y) := \left\{A : X \to Y \,\middle|\, A \textit{ is linear and bounded}\right\}.$$

Then $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ *is a normed vector space. The resulting topology on* $\mathcal{L}(X,Y)$ *is called the* **uniform operator topology**.

Theorem 1.17. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be real normed vector spaces and let* $A : X \to Y$ *be a linear operator. The following are equivalent.*

(i) *A is bounded.*

(ii) *A is continuous.*

(iii) *A is continuous at $x = 0$.*

Proof. We prove that (i) implies (ii). If A is bounded then

$$\|Ax - Ax'\|_Y = \|A(x - x')\|_Y \le \|A\| \, \|x - x'\|_X$$

for all $x, x' \in X$ and so A is Lipschitz-continuous. Since every Lipschitz-continuous function is continuous, this shows that (i) implies (ii). That (ii) implies (iii) follows directly from the definition of continuity.

We prove that (iii) implies (i). Thus assume A is continuous at $x = 0$. Then it follows from the ε-δ definition of continuity with $\varepsilon = 1$ that there exists a constant $\delta > 0$ such that, for all $x \in X$,

$$\|x\|_X < \delta \qquad \Longrightarrow \qquad \|Ax\|_Y < 1.$$

This implies $\|Ax\|_Y \le 1$ for every $x \in X$ with $\|x\|_X = \delta$. Now let $x \in X \setminus \{0\}$. Then $\left\|\delta \|x\|_X^{-1} x\right\|_X = \delta$ and so $\left\|A(\delta \|x\|_X^{-1} x)\right\|_X \le 1$. Multiply both sides of this last inequality by $\delta^{-1} \|x\|_X$ to obtain the inequality

$$\|Ax\|_Y \le \delta^{-1} \|x\|_X \qquad \text{for all } x \in X.$$

This proves Theorem 1.17. ☐

Recall that the **kernel** and **image** of a linear operator $A : X \to Y$ between real vector spaces are the linear subspaces defined by

$$\begin{aligned} \ker(A) &:= \{x \in X \mid Ax = 0\} \subset X, \\ \operatorname{im}(A) &:= \{Ax \mid x \in X\} \subset Y. \end{aligned}$$

If X and Y are normed vector spaces and $A : X \to Y$ is a bounded linear operator, then the kernel of A is a closed subspace of X by Theorem 1.17. However, its image need not be a closed subspace of Y.

Equivalent Norms

Definition 1.18. *Let X be a real vector space. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are called* **equivalent** *if there exists a constant $c \geq 1$ such that*

$$\frac{1}{c}\|x\| \leq \|x\|' \leq c\|x\| \qquad \text{for all } x \in X.$$

Exercise 1.19. (i) This defines an equivalence relation on the set of all norm functions on X.

(ii) Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are equivalent if and only if the identity maps $\mathrm{id} : (X, \|\cdot\|) \to (X, \|\cdot\|')$ and $\mathrm{id} : (X, \|\cdot\|') \to (X, \|\cdot\|)$ are bounded linear operators.

(iii) Two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are equivalent if and only if they induce the same topologies on X, i.e. $\mathscr{U}(X, \|\cdot\|) = \mathscr{U}(X, \|\cdot\|')$.

(iv) Let $\|\cdot\|$ and $\|\cdot\|'$ be equivalent norms on X. Show that $(X, \|\cdot\|)$ is complete if and only if $(X, \|\cdot\|')$ is complete.

1.2.2 Finite-Dimensional Normed Vector Spaces

Theorem 1.20. *Let X be a finite-dimensional real vector space. Then any two norms on X are equivalent.*

Proof. Choose an ordered basis $e_1, \dots, e_n$ on X and define

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2} \qquad \text{for } x = \sum_{i=1}^{n} x_i e_i, \qquad x_i \in \mathbb{R}.$$

This is a norm on X. We prove in two steps that every norm on X is equivalent to $\|\cdot\|_2$. Fix any norm function $X \to \mathbb{R} : x \mapsto \|x\|$.

Step 1. *There is a constant $c > 0$ such that $\|x\| \leq c\|x\|_2$ for all $x \in X$.*

Define $c := \sqrt{\sum_{i=1}^{n} \|e_i\|^2}$ and let $x = \sum_{i=1}^{n} x_i e_i$ with $x_i \in \mathbb{R}$. Then, by the triangle inequality for $\|\cdot\|$ and the Cauchy–Schwarz inequality on $\mathbb{R}^n$, we have

$$\|x\| \leq \sum_{i=1}^{n} |x_i| \|e_i\| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} \sqrt{\sum_{i=1}^{n} \|e_i\|^2} = c\|x\|_2 .$$

This proves Step 1.

Step 2. *There is a constant* $\delta > 0$ *such that* $\delta \|x\|_2 \leq \|x\|$ *for all* $x \in X$.

The set $S := \{x \in X \mid \|x\|_2 = 1\}$ is compact with respect to $\|\cdot\|_2$ by the Heine–Borel Theorem, and the function $S \to \mathbb{R} : x \mapsto \|x\|$ is continuous by Step 1. Hence there is an element $x_0 \in S$ such that $\|x_0\| \leq \|x\|$ for all $x \in S$. Define $\delta := \|x_0\| > 0$. Then every nonzero vector $x \in X$ satisfies $\|x\|_2^{-1} x \in S$, hence $\left\| \|x\|_2^{-1} x \right\| \geq \delta$, and hence $\|x\| \geq \delta \|x\|_2$. This proves Step 2 and Theorem 1.20. □

Theorem 1.20 has several important consequences that are special to finite-dimensional normed vector spaces and do not carry over to infinite dimensions.

Corollary 1.21. *Every finite-dimensional normed vector space is complete.*

Proof. This holds for the Euclidean norm on $\mathbb{R}^n$ by a theorem in first year analysis, which follows rather directly from the completeness of the real numbers. Hence, by Theorem 1.20 and part (iv) of Exercise 1.19, it holds for every norm on $\mathbb{R}^n$. Thus it holds for every finite-dimensional normed vector space. □

Corollary 1.22. *Let* $(X, \|\cdot\|)$ *be a normed vector space. Then every finite-dimensional linear subspace of* X *is a closed subset of* X.

Proof. Let $Y \subset X$ be a finite-dimensional linear subspace and denote by $\|\cdot\|_Y$ the restriction of the norm on X to the subspace Y. Then $(Y, \|\cdot\|_Y)$ is complete by Corollary 1.21 and hence Y is a closed subset of X. □

Corollary 1.23. *Let* $(X, \|\cdot\|)$ *be a finite-dimensional normed vector space and let* $K \subset X$. *Then* K *is compact if and only if* K *is closed and bounded.*

Proof. This holds for the Euclidean norm on $\mathbb{R}^n$ by the Heine–Borel Theorem. Hence it holds for every norm on $\mathbb{R}^n$ by Theorem 1.20. Hence it holds for every finite-dimensional normed vector space. □

Corollary 1.24. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed vector spaces and suppose* $\dim X < \infty$. *Then every linear operator* $A : X \to Y$ *is bounded.*

Proof. Define the function $X \to \mathbb{R} : x \to \|x\|_A$ by

$$\|x\|_A := \|x\|_X + \|Ax\|_Y \qquad \text{for } x \in X.$$

This is a norm on X. Hence, by Theorem 1.20, there exists a constant $c \geq 1$ such that $\|x\|_A \leq c \|x\|_X$ for all $x \in X$. Hence A is bounded. □

The above four corollaries spell out some of the standard facts in finite-dimensional linear algebra. The following four examples show that in none of these four corollaries the hypothesis of finite-dimensionality can be dropped. Thus in functional analysis one must dispense with some of the familiar features of linear algebra. In particular, linear subspaces need no longer be closed subsets and linear maps need no longer be continuous.

Example 1.25. **(i)** Consider the space $X := C([0,1])$ of continuous real valued functions on the closed unit interval $[0,1]$. Then the formulas

$$\|f\|_\infty := \sup_{0\le t\le 1} |f(t)|, \qquad \|f\|_2 := \left(\int_0^1 |f(t)|^2\right)^{1/2}$$

for $f \in C([0,1])$ define norms on X. The space $C([0,1])$ is complete with $\|\cdot\|_\infty$ but not with $\|\cdot\|_2$. Thus the two norms are not equivalent. **Exercise:** Find a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ that is Cauchy with respect to the L^2-norm and has no convergent subsequence.

(ii) The space $Y := C^1([0,1])$ of continuously differentiable real valued functions on the closed unit interval is a dense linear subspace of $C([0,1])$ with the supremum norm and so is not a closed subset of $(C([0,1]), \|\cdot\|_\infty)$.

(iii) Consider the closed unit ball $B := \{f \in C([0,1]) \mid \|f\|_\infty \le 1\}$ in $C([0,1])$ with respect to the supremum norm. This set is closed and bounded, but not equi-continuous. Hence it is not compact by the Arzelà–Ascoli Theorem (see Corollary 1.13). More explicitly, consider the sequence $f_n \in B$ defined by $f_n(t) := \sin(2^n \pi t)$ for $n \in \mathbb{N}$ and $0 \le t \le 1$. It satisfies $\|f_n - f_m\| \ge 1$ for $n \ne m$ and hence does not have any convergent subsequence. More generally, Theorem 1.26 below shows that the compactness of the unit ball characterizes the finite-dimensional normed vector spaces.

(iv) Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and choose an unordered basis $E \subset X$ such that $\|e\| = 1$ for all $e \in E$. Thus every nonzero vector $x \in X$ can be uniquely expressed as a finite linear combination $x = \sum_{i=1}^{\ell} x_i e_i$ with $e_1, \dots, e_\ell \in E$ pairwise distinct and $x_i \in \mathbb{R} \setminus \{0\}$. By assumption E is an infinite set. (The existence of an unordered basis requires the Lemma of Zorn or, equivalently, the axiom of choice by Theorem A.3.) Choose any unbounded function $\lambda : E \to \mathbb{R}$ and define the linear map $\Phi_\lambda : X \to \mathbb{R}$ by $\Phi_\lambda(\sum_{i=1}^{\ell} x_i e_i) := \sum_{i=1}^{\ell} \lambda(e_i) x_i$ for all $\ell \in \mathbb{N}$, all pairwise distinct ℓ-tuples of basis vectors $e_1, \dots, e_\ell \in E$, and all $x_1, \dots, x_\ell \in \mathbb{R}$. Then $\Phi_\lambda : X \to \mathbb{R}$ is an unbounded linear functional.

Theorem 1.26. *Let $(X, \|\cdot\|)$ be a normed vector space and denote the closed unit ball and the closed unit sphere in X by*

$$B := \left\{x \in X \,\middle|\, \|x\| \le 1\right\}, \qquad S := \left\{x \in X \,\middle|\, \|x\| = 1\right\}.$$

Then the following are equivalent.

(i) $\dim X < \infty$.

(ii) *B is compact.*

(iii) *S is compact.*

Proof. That (i) implies (ii) follows from Corollary 1.23 and that (ii) implies (iii) follows from the fact that a closed subset of a compact set in a topological space is compact.

We prove that (iii) implies (i). We argue indirectly and show that if X is infinite-dimensional then S is not compact. Thus assume X is infinite-dimensional. We claim that there exists a sequence $x_i \in X$ such that

$$\|x_i\| = 1, \qquad \|x_i - x_j\| \ge \tfrac{1}{2} \qquad \text{for all } i, j \in \mathbb{N} \text{ with } i \ne j. \tag{1.16}$$

This is then a sequence in S that does not have any convergence subsequence and so it follows that S is not compact.

To prove the existence of a sequence in X satisfying (1.16) we argue by induction and use the axiom of dependent choice. For $i = 1$ choose any element $x_1 \in S$. If $x_1, \dots, x_k \in S$ have been constructed such that $\|x_i - x_j\| \ge \frac{1}{2}$ for $i \ne j$, consider the subspace $Y \subset X$ spanned by the vectors $x_1, \dots, x_k$. This is a closed subspace of X by Corollary 1.22 and is not equal to X because $\dim X = \infty$. Hence Lemma 1.27 below asserts that there exists a vector $x = x_{k+1} \in S$ such that $\|x - y\| \ge \frac{1}{2}$ for all $y \in Y$ and hence, in particular, $\|x_{k+1} - x_i\| \ge \frac{1}{2}$ for $i = 1, \dots, k$. This completes the induction step and shows, by the axiom of dependent choice (see page 10), that there exists a sequence $x_i \in X$ that satisfies (1.16) for $i \ne j$.

More precisely, take

$$\mathbf{X} := \bigsqcup_{k \in \mathbb{N}} S^k$$

and, for every $\mathbf{x} = (x_1, \dots, x_k) \in S^k$, define $\mathbf{A}(\mathbf{x})$ as the set of all $k+1$-tuples $\mathbf{y} = (x_1, \dots, x_k, x) \in S^{k+1}$ such that $\|x - x_i\| \ge \frac{1}{2}$ for $i = 1, \dots, k$. The above argument shows that this set is nonempty for all $\mathbf{x} \in \mathbf{X}$ and so the existence of the required sequence $(x_i)_{i \in \mathbb{N}}$ in S follows from the axiom of dependent choice. This proves Theorem 1.26. □

Lemma 1.27 (Riesz Lemma). *Let $(X, \|\cdot\|)$ be a normed vector space and let $Y \subset X$ be a closed linear subspace that is not equal to X. Fix a constant*

$$0 < \delta < 1.$$

Then there exists a vector $x \in X$ such that

$$\|x\| = 1, \qquad \inf_{y\in Y} \|x - y\| \geq 1 - \delta.$$

Proof. Let $x_0 \in X \setminus Y$. Then

$$d := \inf_{y\in Y} \|x_0 - y\| > 0$$

because Y is closed. Choose $y_0 \in Y$ such that

$$\|x_0 - y_0\| \leq \frac{d}{1-\delta}$$

and define

$$x := \frac{x_0 - y_0}{\|x_0 - y_0\|}.$$

Then $\|x\| = 1$ and

$$\begin{aligned} \|x - y\| &= \frac{\|x_0 - y_0 - \|x_0 - y_0\| \, y\|}{\|x_0 - y_0\|} \\ &\geq \frac{d}{\|x_0 - y_0\|} \\ &\geq 1 - \delta \end{aligned}$$

for all $y \in Y$. This proves Lemma 1.27. □

Theorem 1.26 leads to the question of how one can characterize the compact subsets of an infinite-dimensional Banach space. For the Banach space of continuous functions on a compact metric space with the supremum norm this question is answered by the Arzelà–Ascoli Theorem (Corollary 1.13). The Arzelà–Ascoli Theorem is the source of many other compactness results in functional analysis.