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القسم	الرياضيات
المادة باللغة الانجليزية	functional analysis
المادة باللغة العربية	التحليل الدالي
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Compact Sets

Let (X, d) be a metric space and fix a subset $K \subset X$. Then the restriction of the distance function d to $K \times K$ is a distance function, denoted by $d_K := d|_{K \times K} : K \times K \to \mathbb{R}$, so (K, d_K) is a metric space in its own right. The metric space (X, d) is called **(sequentially) compact** if every sequence in X has a convergent subsequence. The subset K is called **(sequentially) compact** if (K, d_K) is compact, i.e. if every sequence in K has a subsequence that converges to an element of K. It is called **precompact** if its closure is sequentially compact. Thus K is compact if and only if it is precompact and closed. The subset K is called **complete** if (K, d_K) is a complete metric space, i.e. if every Cauchy sequence in K converges to an element of K. It is called **totally bounded** if it is either empty or, for every $\varepsilon > 0$, there exist finitely many elements $\xi_1, \ldots, \xi_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} B_\varepsilon(\xi_i).$$

The next theorem characterizes the compact subsets of a metric space (X, d) in terms of the open subsets of X. It thus shows that compactness depends only on the topology $\mathscr{U}(X, d)$ induced by the distance function d.

Theorem 1.4 (Characterization of Compact Sets). *Let (X, d) be a metric space and let $K \subset X$. Then the following are equivalent.*

(i) *K is sequentially compact.*

(ii) *K is complete and totally bounded.*

(iii) *Every open cover of K has a finite subcover.*

Proof.

Let $(X, \mathscr{U})$ be a topological space. Then condition (iii) in Theorem is used to define compact subsets of X. Thus a subset $K \subset X$ is called **compact** if every open cover of K has a finite subcover. Here an open cover of K is a collection $(U_i)_{i \in I}$ of open subsets $U_i \subset X$, indexed by the elements of a nonempty set I, such that $K \subset \bigcup_{i \in I} U_i$, and a finite subcover is a finite collection of indices $i_1, \ldots, i_m \in I$ such that $K \subset U_{i_1} \cup \cdots \cup U_{i_m}$. Thus Theorem 1.4 asserts that a subset of a metric space (X, d) is sequentially compact if and only if it is compact as a subset of the topological space $(X, \mathscr{U})$ with $\mathscr{U} = \mathscr{U}(X, d)$. A subset of a topological space is called **precompact** if its closure is compact. Elementary properties of compact sets include the fact that every compact subset of a Hausdorff space is closed, that every closed subset of a compact set is compact, and that the image of a compact set under a continuous map is compact

We give two proofs of Theorem 1.4. The first proof is more straight forward and uses the axiom of dependent choice. The second proof is taken from Herrlich [19, Prop 3.26] and only uses the axiom of countable choice.

The **axiom of dependent choice** asserts that, if $\mathbf{X}$ is a nonempty set and $\mathbf{A} : \mathbf{X} \to 2^{\mathbf{X}}$ is a map that assigns to each element $\mathrm{x} \in \mathbf{X}$ a nonempty subset $\mathbf{A}(\mathrm{x}) \subset \mathbf{X}$, then there exists a sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ in $\mathbf{X}$ such that $\mathrm{x}_{k+1} \in \mathbf{A}(\mathrm{x}_k)$ for all $k \in \mathbb{N}$.

In the axiom of dependent choice the first element of the sequence $(\mathrm{x}_k)_{k\in\mathbb{N}}$ can be prescribed. To see this, fix an element $\mathrm{x}_1 \in \mathbf{X}$, define $\widetilde{\mathbf{X}}$ as the set of all tuples of the form $\widetilde{\mathrm{x}} = (n, \mathrm{x}_1, \dots, \mathrm{x}_n)$ with $n \in \mathbb{N}$ and $\mathrm{x}_k \in \mathbf{A}(\mathrm{x}_{k-1})$ for $k = 2, \dots, n$, and define $\widetilde{\mathbf{A}}(\widetilde{\mathrm{x}}) := \{(n+1, \mathrm{x}_1, \dots, \mathrm{x}_n, \mathrm{x}) \,|\, \mathrm{x} \in \mathbf{A}(\mathrm{x}_n)\}$ for $\widetilde{\mathrm{x}} = (n, \mathrm{x}_1, \dots, \mathrm{x}_n) \in \widetilde{\mathbf{X}}$. Then $\widetilde{\mathbf{X}}$ is nonempty and $\widetilde{\mathbf{A}}(\widetilde{\mathrm{x}})$ is nonempty for every $\widetilde{\mathrm{x}} \in \widetilde{\mathbf{X}}$. Now apply the axiom of dependent choice to $\widetilde{\mathbf{A}}$.

The **axiom of countable choice** asserts that, if $(\mathbf{A}_k)_{k\in\mathbb{N}}$ is a sequence of nonempty subsets of a set $\mathbf{A}$, then there exists a sequence $(\mathrm{a}_k)_{k\in\mathbb{N}}$ in $\mathbf{A}$ such that $\mathrm{a}_k \in \mathbf{A}_k$ for all $k \in \mathbb{N}$. It follows from the axiom of dependent choice by taking $\mathbf{X} := \mathbb{N} \times \mathbf{A}$ and $\mathbf{A}(k, \mathrm{a}) := \{k+1\} \times \mathbf{A}_{k+1}$ for $(k, \mathrm{a}) \in \mathbb{N} \times \mathbf{A}$.

Lemma 1.5. *Let (X, d) be a metric space and let $K \subset X$. Then the following are equivalent.*

(i) *Every sequence in K has a Cauchy subsequence.*

(ii) *K is totally bounded.*

Proof. We prove that (ii) implies (i). Thus assume that K is totally bounded and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in K. We prove that there exists a sequence of infinite subsets $\mathbb{N} \supset T_1 \supset T_2 \supset \cdots$ such that, for all $k, m, n \in \mathbb{N}$,

$$m, n \in T_k \qquad \Longrightarrow \qquad d(x_m, x_n) < 2^{-k}. \tag{1.5}$$

Since K is totally bounded, it follows from the axiom of countable choice that there exists a sequence of ordered finite subsets $S_k = \{\xi_{k,1}, \dots, \xi_{k,m_k}\} \subset K$ such that $K \subset \bigcup_{i=1}^{m_k} B_{2^{-k-1}}(\xi_{k,i})$ for all $k \in \mathbb{N}$. Since $x_n \in K$ for all $n \in \mathbb{N}$, there must exist an index $i \in \{1, \dots, m_1\}$ such that the open ball $B_{1/4}(\xi_{1,i})$ contains infinitely many of the elements x_n. Let i_1 be the smallest such index and define the set

$$T_1 := \{n \in \mathbb{N} \,|\, x_n \in B_{1/4}(\xi_{1,i_1})\}.$$

This set is infinite and satisfies $d(x_n, x_m) \le d(x_n, \xi_{1,i_1}) + d(\xi_{1,i_1}, x_m) < 1/2$ for all $m, n \in T_1$. Now fix an integer $k \ge 2$ and suppose, by induction, that T_{k-1} has been defined. Since T_{k-1} is an infinite set, there must exist an index $i \in \{1, \dots, m_k\}$ such that the ball $B_{2^{-k-1}}(\xi_{k,i})$ contains infinitely many of the elements x_n with $n \in T_{k-1}$. Let i_k be the smallest such index and define

$$T_k := \{ n \in T_{k-1} \,|\, x_n \in B_{2^{-k-1}}(\xi_{k,i_k}) \}.$$

This set is infinite and satisfies $d(x_n, x_m) \le d(x_n, \xi_{k,i_k}) + d(\xi_{k,i_k}, x_m) < 2^{-k}$ for all $m, n \in T_k$. This completes the induction argument and the construction of a decreasing sequence of infinite sets $T_k \subset \mathbb{N}$ that satisfy (1.5).

We prove that $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence. By (1.5) there exists a sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ such that $n_k \in T_k$ for all $k \in \mathbb{N}$. Such a sequence can be defined by the recursion formula

$$n_1 := \min T_1, \qquad n_{k+1} := \min \{ n \in T_k \,|\, n > n_k \}$$

for $k \in \mathbb{N}$. It follows that $n_k, n_\ell \in T_k$ and hence

$$d(x_{n_k}, x_{n_\ell}) < 2^{-k} \qquad \text{for } \ell \ge k \ge 1.$$

Thus the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence.

We prove that (i) implies (ii), following [19, Prop 3.26]. We argue indirectly and assume that K is not totally bounded and hence also nonempty. Then there exists a constant $\varepsilon > 0$ such that K does not admit a finite cover by balls of radius ε, centered at elements of K. We prove in three steps that there exists a sequence in K that does not have a Cauchy subsequence.

Step 1. *For $n \in \mathbb{N}$ define the set*

$$K_n := \left\{ (x_1, \dots, x_n) \in K^n \,\middle|\, \begin{array}{l} \text{if } i, j \in \{1, \dots, n\} \text{ and } i \ne j \\ \text{then } d(x_i, x_j) \ge \varepsilon \end{array} \right\}.$$

There is a sequence $(x_k)_{k\in\mathbb{N}}$ in K such that $(x_{n(n-1)/2+1}, \dots, x_{n(n+1)/2}) \in K_n$ for every integer $n \ge 1$.

We prove that K_n is nonempty for every $n \in \mathbb{N}$. For $n = 1$ this holds because K is nonempty. If it is empty for some $n \in \mathbb{N}$ then there exists an integer $n \ge 1$ such that $K_n \ne \emptyset$ and $K_{n+1} = \emptyset$. In this case, choose an element $(x_1, \dots, x_n) \in K_n$. Since $K_{n+1} = \emptyset$, this implies $K \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$, contradicting the choice of ε. Since $K_n \ne \emptyset$ for all $n \in \mathbb{N}$, the existence of a sequence $(x_k)_{k\in\mathbb{N}}$ as in Step 1 follows from the axiom of countable choice.

Step 2. *For every collection of $n-1$ elements $y_1, \dots, y_{n-1} \in K$, there is an integer i such that $\frac{(n-1)n}{2} < i \le \frac{n(n+1)}{2}$ and $d(y_j, x_i) \ge \frac{\varepsilon}{2}$ for $j = 1, \dots, n-1$.*

Otherwise, there exists a map $\nu : \{\frac{(n-1)n}{2} + 1, \dots, \frac{n(n+1)}{2}\} \to \{1, \dots, n-1\}$ such that $d(x_i, y_{\nu(i)}) < \frac{\varepsilon}{2}$ for all i. Since the target space of ν has smaller cardinality than the domain, there is a pair $i \ne j$ in the domain with $\nu(i) = \nu(j)$ and so $d(x_i, x_j) \le d(x_i, y_{\nu(i)}) + d(y_{\nu(j)}, x_j) < \varepsilon$, in contradiction to Step 1.

Step 3. *There exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $k_1 = 1$ and*

$$\frac{(n-1)n}{2} < k_n \le \frac{n(n+1)}{2}, \qquad d(x_{k_m}, x_{k_n}) \ge \frac{\varepsilon}{2} \qquad \textit{for } m < n. \tag{1.6}$$

Define $k_1 := 1$, fix an integer $n \ge 2$, and assume, by induction, that the integers $k_1, k_2, \dots, k_{n-1}$ have been found such that (1.6) holds with n replaced by any number $n' \in \{2, \dots, n-1\}$. Then, by Step 2, there exists a unique smallest integer k_n such that $\frac{(n-1)n}{2} < k_n \le \frac{n(n+1)}{2}$ and $d(x_{k_m}, x_{k_n}) \ge \frac{\varepsilon}{2}$ for $m = 1, \dots, n-1$. This proves the existence of a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ that satisfies (1.6).

The sequence $(x_{k_n})_{n\in\mathbb{N}}$ in Step 3 does not have a Cauchy subsequence. This shows that (i) implies (ii) and completes the proof of Lemma 1.5. □

First proof of Theorem 1.4. We prove that (i) implies (iii). Assume that K is nonempty and sequentially compact and let $\{U_i\}_{i\in I}$ be an open cover of K. Here I is a nonempty index set and the map $I \to 2^X : i \mapsto U_i$ assigns to each index i an open set $U_i \subset X$ such that $K \subset \bigcup_{i\in I} U_i$. We prove in two steps that there exist finitely many indices $i_1, \dots, i_m \in I$ such that $K \subset \bigcup_{j=1}^m U_{i_j}$.

Step 1. *There exists a constant $\varepsilon > 0$ such that, for every $x \in K$, there exists an index $i \in I$ such that $B_\varepsilon(x) \subset U_i$.*

Assume, by contradiction, that there is no such constant $\varepsilon > 0$. Then

$$\forall \varepsilon > 0 \;\; \exists x \in K \;\; \forall i \in I \;\; B_\varepsilon(x) \not\subset U_i.$$

Take $\varepsilon = 1/n$ for $n \in \mathbb{N}$. Then the set $\{x \in K \,|\, B_{1/n}(x) \not\subset U_i \text{ for all } i \in I\}$ is nonempty for every $n \in \mathbb{N}$. Hence the axiom of countable choice asserts that there exists a sequence $x_n \in K$ such that

$$B_{1/n}(x_n) \not\subset U_i \qquad \text{for all } n \in \mathbb{N} \text{ and all } i \in I. \tag{1.7}$$

Since K is sequentially compact, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to an element $x \in K$. Since $K \subset \bigcup_{i\in I} U_i$, there exists an $i \in I$ such that $x \in U_i$. Since U_i is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U_i$. Since $x = \lim_{k\to\infty} x_{n_k}$, there is a $k \in \mathbb{N}$ such that $d(x, x_{n_k}) < \frac{\varepsilon}{2}$ and $\frac{1}{n_k} < \frac{\varepsilon}{2}$. Thus $B_{1/n_k}(x_{n_k}) \subset B_{\varepsilon/2}(x_{n_k}) \subset B_\varepsilon(x) \subset U_i$ in contradiction to (1.7).

Step 2. *There exist indices $i_1, \dots, i_m \in I$ such that $K \subset \bigcup_{j=1}^m U_{i_j}$.*

Assume, by contradiction, that this is wrong. Let $\varepsilon > 0$ be the constant in Step 1. We prove that there are sequences $x_n \in K$ and $i_n \in I$ such that

$$B_\varepsilon(x_n) \subset U_{i_n}, \qquad x_n \notin U_{i_1} \cup \cdots \cup U_{i_{n-1}} \tag{1.8}$$

for all $n \in \mathbb{N}$ (with $n \geq 2$ for the second condition). Choose $x_1 \in K$. Then, by Step 1, there exists an index $i_1 \in I$ such that $B_\varepsilon(x_1) \subset U_{i_1}$. Now suppose, by induction, that $x_1, \ldots, x_k$ and $i_1, \ldots, i_k$ have been found such that (1.8) holds for $n \leq k$. Then $K \not\subset U_{i_1} \cup \cdots \cup U_{i_k}$. Choose an element $x_{k+1} \in K \setminus (U_{i_1} \cup \cdots \cup U_{i_k})$. By Step 1, there exists an index $i_{k+1} \in I$ such that $B_\varepsilon(x_{k+1}) \subset U_{i_{k+1}}$. Thus the existence of sequences x_n and i_n that satisfy (1.8) follows from the axiom of dependent choice.

By (1.8) we have $d(x_n, x_k) \geq \varepsilon$ for $k \neq n$, so $(x_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence, contradicting (i). This shows that (i) implies (iii).

We prove that (iii) implies (ii). Thus assume that every open cover of K has a finite subcover. Assume that K is nonempty and fix a constant $\varepsilon > 0$. Then the sets $B_\varepsilon(\xi)$ for $\xi \in K$ form a nonempty open cover of K. Hence there exist finitely many elements $\xi_1, \ldots, \xi_m \in K$ such that $K \subset \bigcup_{i=1}^m B_\varepsilon(\xi_i)$. This shows that K is totally bounded.

We prove that K is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in K and suppose, by contradiction, that $(x_n)_{n\in\mathbb{N}}$ does not converge to any element of K. Then no subsequence of $(x_n)_{n\in\mathbb{N}}$ can converge to any element of K. Thus, for every $\xi \in K$, there is an $\varepsilon > 0$ such that $B_\varepsilon(\xi)$ contains only finitely many of the x_n. For $\xi \in K$ let $\varepsilon(\xi) > 0$ be half the supremum of the set of all $\varepsilon \in (0, 1]$ such that $\#\{n \in \mathbb{N} \,|\, x_n \in B_\varepsilon(\xi)\} < \infty$. Then the set $\{n \in \mathbb{N} \,\big|\, x_n \in B_{\varepsilon(\xi)}(\xi)\}$ is finite for every $\xi \in K$. Thus $\{B_{\varepsilon(\xi)}(\xi)\}_{\xi\in K}$ is an open over of K that does not have a finite subcover, in contradiction to (iii). This shows that (iii) implies (ii).

That (ii) implies (i) follows from Lemma 1.5 and this completes the first proof of Theorem 1.4

The above proof of Theorem 1.4 requires the axiom of dependent choice and only uses the implication (ii) $\Longrightarrow$ (i) in Lemma 1.5. The second proof follows [19 Prop 3.26]. It only requires the axiom of countable choice, but uses both directions in Lemma

Second Proof of Theorem 1.4 Every sequentially compact metric space is complete, because a Cauchy sequence converges if and only if it has a convergent subsequence. Hence the equivalence of (i) and (ii) in Theorem follows directly from Lemma 1.5.

We prove that (ii) implies (iii). Assume that K is complete and totally bounded. Suppose, by contradiction, that there is an open cover $\{U_i\}_{i\in I}$ of K that does not have a finite subcover. Then $K \neq \emptyset$. For $n, m \in \mathbb{N}$ define

$$A_{n,m} := \left\{ (x_1, \ldots, x_m) \in K^m \,\middle|\, K \subset \bigcup_{j=1}^m B_{1/n}(x_j) \right\}.$$

Then, for every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $A_{n,m} \neq \emptyset$, because K is totally bounded and nonempty. For $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be the smallest positive integer such that $A_{n,m_n} \neq \emptyset$. Then, by the axiom of countable choice, there is a sequence

$$a_n = (x_{n,1}, \ldots, x_{n,m_n}) \in A_{n,m_n} \qquad \text{for } n \in \mathbb{N}.$$

Next we construct a sequence $(y_n)_{n\in\mathbb{N}}$ in K such that the intersection

$$\bigcap_{\nu=1}^{n} B_{1/\nu}(y_\nu) \cap K$$

cannot be covered by finitely many of the sets U_i for any $n \in \mathbb{N}$. For $n = 1$ define $y_1 := x_{1,k}$, where

$$k := \min \left\{ j \in \{1, \ldots, m_1\} \;\middle|\; \begin{array}{l} \text{the set } B_1(x_{1,j}) \cap K \text{ cannot} \\ \text{be covered by finitely many } U_i \end{array} \right\}.$$

Assume, by induction, that $y_1, \ldots, y_{n-1}$ have been chosen such that the set $\bigcap_{\nu=1}^{n-1} B_{1/\nu}(y_\nu) \cap K$ cannot be covered by finitely many of the U_i and define $y_n := x_{n,k}$, where

$$k := \min \left\{ j \in \{1, \ldots, m_n\} \;\middle|\; \begin{array}{l} \text{the set } B_{1/n}(x_{n,j}) \cap \bigcap_{\nu=1}^{n-1} B_{1/\nu}(y_\nu) \cap K \\ \text{cannot be covered by finitely many } U_i \end{array} \right\}.$$

This completes the construction of the sequence $(y_n)_{n\in\mathbb{N}}$. It satisfies

$$d(y_n, y_m) < \frac{1}{m} + \frac{1}{n} \leq \frac{2}{m} \qquad \text{for } n > m \geq 1,$$

because $B_{1/n}(y_n) \cap B_{1/m}(y_m) \neq \emptyset$. Hence $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in K. Since K is complete, the limit

$$y^* := \lim_{n\to\infty} y_n$$

exists and is an element of K. Choose an index $i^* \in I$ such that $y^* \in U_{i^*}$ and choose a constant $\varepsilon^* > 0$ such that

$$B_{\varepsilon^*}(y^*) \subset U_{i^*}.$$

Then

$$B_{1/n}(y_n) \subset B_{\varepsilon^*}(y^*) \subset U_{i^*}$$

$$B_{1/n}(y_n) \subset B_{\varepsilon^*}(y^*) \subset U_{i^*}$$

for n sufficiently large in contradiction to the choice of y_n. This proves that (ii) implies (iii).

That (iii) implies (ii) was shown in the first proof without using any version of the axiom of choice. This completes the second proof of Theorem 1.4

It follows immediately from Theorem 1.4 that every compact metric space is separable. Here are the relevant definitions.

Definition 1.6. *Let X be a topological space. A subset $S \subset X$ is called* **dense in** X *if its closure is equal to X or, equivalently, every nonempty open subset of X contains an element of S. The space X is called* **separable** *if it admits a countable dense subset. (A set is called* **countable** *if it is either finite or countably infinite.)*

Corollary 1.7. *Every compact metric space is separable.*

Proof. Let $n \in \mathbb{N}$. Since X is totally bounded by Theorem 1.4 there exists a finite set $S_n \subset X$ such that $X = \bigcup_{\xi \in S_n} B_{1/n}(\xi)$. Hence $S := \bigcup_{n \in \mathbb{N}} S_n$ is a countable dense subset of X.

Corollary 1.8. *Let (X, d) be a metric space and let $A \subset X$. Then the following are equivalent.*

(i) *A is precompact.*

(ii) *Every sequence in A has a subsequence that converges in X.*

(iii) *A is totally bounded and every Cauchy sequence in A converges in X.*

Proof. That (i) implies (ii) follows directly from the definitions.

We prove that (ii) implies (iii). By (ii) every sequence in A has a Cauchy subsequence and so A is totally bounded by Lemma 1.5 If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in A, then by (ii) there exists a subsequence $(x_{n_i})_{i \in \mathbb{N}}$ that converges in X, and so the original sequence converges in X because a Cauchy sequence converges if and only if it has a convergent subsequence.

We prove that (iii) implies (i). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in the closure $\overline{A}$ of A. Then, by the axiom of countable choice, there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in A such that $d(x_n, a_n) < 1/n$ for all $n \in \mathbb{N}$. Since A is totally bounded, it follows from Lemma 1.5 that the sequence $(a_n)_{n \in \mathbb{N}}$ has a Cauchy subsequence $(a_{n_i})_{i \in \mathbb{N}}$. This subsequence converges in X by (iii). Denote its limit by a. Then $a \in \overline{A}$ and $a = \lim_{i \to \infty} a_{n_i} = \lim_{i \to \infty} x_{n_i}$. Thus $\overline{A}$ is sequentially compact. This proves Corollary 1.8

Corollary 1.9. *Let (X,d) be a complete metric space and let $A \subset X$. Then the following are equivalent.*

(i) *A is precompact.*

(ii) *Every sequence in A has a Cauchy subsequence.*

(iii) *A is totally bounded.*

Proof. This follows directly from the definitions and Corollary 1.8.

1.1.3 The Arzelà–Ascoli Theorem

It is a recurring theme in functional analysis to understand which subsets of a Banach space or topological vector space are compact. For the standard Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$ the Heine-Borel Theorem asserts that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. This continues to hold for every finite-dimensional normed vector space and, conversely, every normed vector space in which the closed unit ball is compact is necessarily finite-dimensional (see Theorem 1.26 below). For infinite-dimensional Banach spaces this leads to the problem of characterizing the compact subsets. Necessary conditions are that the subset is closed and bounded, however, these conditions can no longer be sufficient. For the Banach space of continuous functions on a compact metric space a characterization of the compact subsets is given by a theorem of Arzelà and Ascoli which we explain next.

Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Then the space

$$C(X,Y) := \{f : X \to Y \mid f \text{ is continuous}\}$$

of continuous maps from X to Y is a metric space with the distance function

$$d(f,g) := \sup_{x\in X} d_Y(f(x), g(x)) \qquad \text{for } f,g \in C(X,Y). \tag{1.9}$$

This is well defined because the function $X \to \mathbb{R} : x \mapsto d_Y(f(x), g(x))$ is continuous and hence is bounded because X is compact. That (1.9) satisfies the axioms of a distance function follows directly from the definitions. When X is nonempty, the metric space $C(X,Y)$ with the distance function (1.9) is complete if and only if Y is complete, because the limit of a uniformly convergent sequence of continuous functions is again continuous.

Definition 1.10. *A subset $\mathscr{F} \subset C(X,Y)$ is called* **equi-continuous** *if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $x, x' \in X$ and all $f \in \mathscr{F}$,*

$$d_X(x,x') < \delta \qquad \Longrightarrow \qquad d_Y(f(x), f(x')) < \varepsilon.$$

It is called **pointwise compact** *if, for every $x \in X$, the set*

$$\mathscr{F}(x) := \{f(x) \mid f \in \mathscr{F}\}$$

is a compact subset of Y. It is called **pointwise precompact** *if, for every $x \in X$, the set $\mathscr{F}(x)$ has a compact closure in Y.*

Since every continuous map defined on a compact metric space is uniformly continuous, every finite subset of $C(X,Y)$ is equi-continuous.

Theorem 1.11 (Arzelà–Ascoli). *Let (X, d_X) be a compact metric space, let (Y, d_Y) be a metric space, and let $\mathscr{F} \subset C(X,Y)$. Then the following are equivalent.*

(i) *$\mathscr{F}$ is precompact.*

(ii) *$\mathscr{F}$ is pointwise precompact and equi-continuous.*

Proof. We prove that (i) implies (ii). Thus assume $\mathscr{F}$ is precompact. That $\mathscr{F}$ is pointwise precompact follows from the fact that the evaluation map

$$C(X,Y) \to Y : f \mapsto \mathrm{ev}_x(f) := f(x)$$

is continuous for every $x \in X$. Since the image of a precompact set under a continuous map is again precompact (Exercise 1.60), it follows that the set $\mathscr{F}(x) = \mathrm{ev}_x(\mathscr{F})$ is a precompact subset of Y for every $x \subset X$.

It remains to prove that $\mathscr{F}$ is equi-continuous. Assume $\mathscr{F}$ is nonempty and fix a constant $\varepsilon > 0$. Since the set $\mathscr{F}$ is totally bounded by Lemma 1.5, there exist finitely many maps $f_1, \dots, f_m \in \mathscr{F}$ such that

$$\mathscr{F} \subset \bigcup_{i=1}^{m} B_{\varepsilon/3}(f_i). \tag{1.10}$$

Since X is compact, each function f_i is uniformly continuous. Hence there exists a constant $\delta > 0$ such that, for all $i \in \{1, \dots, m\}$ and all $x, x' \in X$,

$$d_X(x, x') < \delta \qquad \Longrightarrow \qquad d_Y(f_i(x), f_i(x')) < \varepsilon/3. \tag{1.11}$$

Now let $f \in \mathscr{F}$ and let $x, x' \in X$ such that $d_X(x, x') < \delta$. Then it follows from (1.10) that there is an index $i \in \{1, \dots, m\}$ such that $d(f, f_i) < \varepsilon/3$. Thus $d_Y(f(x), f_i(x)) < \varepsilon/3$ and $d_Y(f(x'), f_i(x')) < \varepsilon/3$. Moreover, it follows from (1.11) that $d_Y(f_i(x), f_i(x')) < \varepsilon/3$. Hence, by the triangle inequality,

$$\begin{aligned} d_Y(f(x), f(x')) &\le d_Y(f(x), f_i(x)) + d_Y(f_i(x), f_i(x')) + d_Y(f_i(x'), f(x')) \\ &\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This shows that $\mathscr{F}$ is equi-continuous, We prove that (ii) implies (i). Assume X and Y are nonempty, let $(x_k)_{k\in\mathbb{N}}$ be a dense sequence in X (Corollary 1.7), and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. We prove in three steps that $(f_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Step 1. *There exists a subsequence $(g_i)_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that the sequence $(g_i(x_k))_{i\in\mathbb{N}}$ converges in Y for every $k \in \mathbb{N}$.*

Since $\mathscr{F}(x_k)$ is precompact for each k, it follows from the axiom of dependent choice (page 10) that there is a sequence of subsequences $(f_{n_{k,i}})_{i\in\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, the sequence $(f_{n_{k+1,i}})_{i\in\mathbb{N}}$ is a subsequence of $(f_{n_{k,i}})_{i\in\mathbb{N}}$ and the sequence $(f_{n_{k,i}}(x_k))_{i\in\mathbb{N}}$ converges in Y. Thus the diagonal subsequence $g_i := f_{n_{i,i}}$ satisfies the requirements of Step 1.

Step 2. *Let g_i be as in Step 1. Then $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C(X, Y)$.*

Fix a constant $\varepsilon > 0$. Then, by equi-continuity, there exists a constant $\delta > 0$ such that, for all $f \in \mathscr{F}$ and all $x, x' \in X$,

$$d_X(x, x') < \delta \qquad \Longrightarrow \qquad d_Y(f(x), f(x')) < \varepsilon/3. \tag{1.12}$$

Since the balls $B_\delta(x_k)$ form an open cover of X, there exists an $m \in \mathbb{N}$ such that $X = \bigcup_{k=1}^m B_\delta(x_k)$. Since $(g_i(x_k))_{i\in\mathbb{N}}$ is a Cauchy sequence for each k, there exists an $N \in \mathbb{N}$ such that, for all $i, j, k \in \mathbb{N}$, we have

$$1 \le k \le m, \quad i, j \ge N \qquad \Longrightarrow \qquad d_Y(g_i(x_k), g_j(x_k)) < \varepsilon/3. \tag{1.13}$$

We prove that $d(g_i, g_j) < \varepsilon$ for all $i, j \ge N$. To see this, fix an element $x \in X$. Then there exists an index $k \in \{1, \dots, m\}$ such that $d_X(x, x_k) < \delta$. This implies $d_Y(g_i(x), g_i(x_k)) < \varepsilon/3$ for all $i \in \mathbb{N}$, by (1.12), and so

$$\begin{aligned} d_Y(g_i(x), g_j(x)) &\le d_Y(g_i(x), g_i(x_k)) + d_Y(g_i(x_k), g_j(x_k)) + d_Y(g_j(x_k), g_j(x)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for all $i, j \ge N$ by (1.13). Hence $d(g_i, g_j) = \max_{x\in X} d_Y(g_i(x), g_j(x)) < \varepsilon$ for all $i, j \ge N$ and this proves Step 2.

Step 3. *The subsequence $(g_i)_{i\in\mathbb{N}}$ in Step 1 converges in $C(X, Y)$.*

Let $x \in X$. By Step 2, $(g_i(x))_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{F}(x)$. Since $\mathscr{F}(x)$ is a precompact subset of Y, the sequence $(g_i(x))_{i\in\mathbb{N}}$ has a convergent subsequence and hence converges in Y. Denote the limit by $g(x) := \lim_{i\to\infty} g_i(x)$. Then the sequence g_i converges uniformly to g by Step 2 and so $g \in C(X, Y)$.

Step 3 shows that every sequence in $\mathscr{F}$ has a subsequence that converges to an element of $C(X, Y)$. Hence $\mathscr{F}$ is precompact by Corollary 1.8. This proves Theorem 1.11. □

Corollary 1.12 (Arzelà–Ascoli). *Let (X, d_X) be a compact metric space, let (Y, d_Y) be a metric space, and let $\mathscr{F} \subset C(X, Y)$. Then the following are equivalent.*

(i) *$\mathscr{F}$ is compact.*

(ii) *$\mathscr{F}$ is closed, pointwise compact, and equi-continuous.*

(iii) *$\mathscr{F}$ is closed, pointwise precompact, and equi-continuous.*

Proof. That (i) implies (ii) follows from Theorem 1.11, because every compact subset of a metric space is closed, and the image of a compact set under a continuous map is compact. Here the continuous map in question is the evaluation map $C(X, Y) \to Y : f \mapsto f(x)$ associated to $x \in X$. That (ii) implies (iii) is obvious. That (iii) implies (i) follows from Theorem 1.11, because a subset of a metric space is compact if and only if it is precompact and closed. This proves Corollary 1.12. □

When the target space Y is the Euclidean space $(\mathbb{R}^n, \|\cdot\|_2)$ in part (i) of Example 1.3, the Arzelà–Ascoli Theorem takes the following form.

Corollary 1.13 (Arzelà–Ascoli). *Let (X, d) be a compact metric space and let $\mathscr{F} \subset C(X, \mathbb{R}^n)$. Then the following holds.*

(i) *$\mathscr{F}$ is precompact if and only if it is bounded and equi-continuous.*

(ii) *$\mathscr{F}$ is compact if and only if it is closed, bounded, and equi-continuous.*

Proof. Assume $\mathscr{F}$ is precompact. Then $\mathscr{F}$ is equi-continuous by Theorem 1.11, and is bounded, because a sequence whose norm tends to infinity cannot have a convergent subsequence. Conversely, assume $\mathscr{F}$ is bounded and equi-continuous. Then, for each $x \in X$, the set $\mathscr{F}(x) \subset \mathbb{R}^n$ is bounded and therefore is precompact by the Heine–Borel Theorem. Hence $\mathscr{F}$ is precompact by Theorem 1.11. This proves (i). Part (ii) follows from (i) and the fact that a subset of a metric space is compact if and only if it is precompact and closed. This proves Corollary 1.13. □

Exercise 1.14. This exercise shows that the hypothesis that X is compact cannot be removed in Corollary 1.13. Consider the Banach space $C_b(\mathbb{R})$ of bounded continuous real-valued functions on $\mathbb{R}$ with the supremum norm. Find a closed bounded equi-continuous subset of $C_b(\mathbb{R})$ that is not compact.

There are many versions of the Arzelà–Ascoli Theorem. For example, Theorem 1.11, Corollary 1.12, and Corllary 1.13 continue to hold, with the appropriate notion of equi-continuity, when X is any compact topological space. This is the content of the following exercise.

Exercise 1.15. Let X be a compact topological space and let Y be a metric space. Then the space $C(X, Y)$ of continuous functions $f : X \to Y$ is a metric space with the distance function (1.9). A subset $\mathscr{F} \subset C(X, Y)$ is called **equi-continuous** if, for every $x \in X$ and every $\varepsilon > 0$, there exists an open neighborhood $U \subset X$ of x such that $d_Y(f(x), f(x')) < \varepsilon$ for all $x' \in U$ and all $f \in \mathscr{F}$.

(a) Prove that the above definition of equi-continuity agrees with the one in Definition 1.10 whenever X is a compact metric space.

(b) Prove the following variant of the Arzelà–Ascoli Theorem for compact topological spaces X.

Arzelà–Ascoli Theorem. *Let X be a compact topological space and let Y be a metric space. A set $\mathscr{F} \subset C(X, Y)$ is precompact if and only if it is pointwise precompact and equi-continuous.*

Hint 1: If $\mathscr{F}$ is precompact, use the argument in the proof of Theorem 1.11 to show that $\mathscr{F}$ is pointwise precompact and equi-continuous.

Hint 2: Assume $\mathscr{F}$ is equi-continuous and pointwise precompact.

Step 1. The set $F := \{f(x) \,|\, x \in X,\, f \in \mathscr{F}\} \subset Y$ is totally bounded.
Show that F is precompact (Exercise 1.60) and use Corollary 1.8.

Step 2. The set $\mathscr{F}$ is totally bounded.
Let $\varepsilon > 0$. Cover F by finitely many open balls $V_1, \dots, V_n$ of radius $\varepsilon/3$ and cover X by finitely many open sets $U_1, \dots, U_m$ such that

$$\sup_{x,x' \in U_i} \sup_{f \in \mathscr{F}} d_Y(f(x), f(x')) < \varepsilon/3 \qquad \text{for } i = 1, \dots, m.$$

For any function $\alpha : \{1, \dots, m\} \to \{1, \dots, n\}$ define

$$\mathscr{F}_\alpha := \left\{ f \in \mathscr{F} \,\middle|\, f(U_i) \cap V_{\alpha(i)} \neq \emptyset \text{ for } i = 1, \dots, m \right\}.$$

Prove that $d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) < \varepsilon$ for all $f, g \in \mathscr{F}_\alpha$. Let A be the set of all α such that $\mathscr{F}_\alpha \neq \emptyset$. Prove that $\mathscr{F} = \bigcup_{\alpha \in A} \mathscr{F}_\alpha$ and choose a collection of functions $f_\alpha \in \mathscr{F}_\alpha$, one for each $\alpha \in A$.

Step 3. The set $\mathscr{F}$ is precompact.
Use Lemma 1.5 and Step 3 in the proof of Theorem 1.11 to show that every sequence in $\mathscr{F}$ has a subsequence that converges in $C(X, Y)$.

In contrast to what one might expect from Exercise 1.14, there is also a version of the Arzelà–Ascoli theorem for the space of continuous functions from an arbitrary topological space X to a metric space Y. This version uses the compact-open topology on $C(X, Y)$ and is explained in Exercise 3.63.