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Metric Spaces and Compact Sets

Definition: A metric space is a pair (X, d) consisting of a set X and a function

$$d: XxX \to \mathbb{R}$$

that satisfies the following axioms.

(M1) $d(x, y) \geq 0$ *for all* $x, y \in X$, *with equality if and only if* $x = y$.

(M2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(M3) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

A function $d : X \times X \to \mathbb{R}$ *that satisfies these axioms is called a* **distance function** *and the inequality in (M3) is called the* **triangle inequality**. *A subset* $U \subset X$ *of a metric space* (X, d) *is called* **open** *(or* d-**open***) if, for every* $x \in U$, *there exists a constant* $\varepsilon > 0$ *such that the* **open ball**

$$B_\varepsilon(x) := B_\varepsilon(x, d) := \{y \in X \mid d(x, y) < \varepsilon\}$$

(centered at x *with radius* ε*) is contained in* U. *The set of* d*-open subsets of* X *will be denoted by*

$$\mathscr{U}(X, d) := \{U \subset X \mid U \text{ is } d\text{-open}\}.$$

It follows directly from the definitions that the collection $\mathscr{U}(X, d) \subset 2^X$ of d-open sets in a metric space (X, d) satisfies the axioms of a **topology** (i.e. the empty set and the set X are open, arbitrary unions of open sets are open, and finite intersections of open sets are open). A subset F of a metric space (X, d) is closed (i.e. its complement is open) if and only if the limit point of every convergent sequence in F is itself contained in F.

Recall that a **Cauchy sequence** in a metric space (X, d) is a sequence $(x_n)_{n \in \mathbb{N}}$ with the property that, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that any two integers $n, m \geq n_0$ satisfy the inequality $d(x_n, x_m) < \varepsilon$. Recall also that a metric space (X, d) is called **complete** if every Cauchy sequence in X converges.

The most important metric spaces in the field of functional analysis are the normed vector spaces.

Definition: (Banach space). A normed vector space is a pair

$(X, \|\cdot\|)$ consisting of a real vector space X and a function $X \to \mathbb{R} : x \mapsto \|x\|$ satisfying the following.

(N1) *$\|x\| \geq 0$ for all $x \in X$, with equality if and only if $x = 0$.*

(N2) *$\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.*

(N3) *$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.*

Let $(X, \|\cdot\|)$ be a normed vector space. Then the formula

$$d(x, y) := \|x - y\|$$

for $x, y \in X$ defines a distance function on X. The resulting topology is denoted by $\mathscr{U}(X, \|\cdot\|) := \mathscr{U}(X, d)$. X is called a **Banach space** *if the metric space (X, d) is* **complete**, *i.e. if every Cauchy sequence in X converges.*

Here are six examples of Banach spaces.

Example (i) Fix a real number $1 \leq p < \infty$. Then the vector space $\mathbb{R}^n$ of all n-tuples $x = (x_1, \ldots, x_n)$ of real numbers is a Banach space with the norm-function

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $p = 2$ this is the Euclidean norm. Another norm is given by $\|x\|_\infty := \max_{i=1,\ldots,n} |x_i|$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

(ii) For $1 \leq p < \infty$ the set of p-summable sequences of real numbers is denoted by

$$\ell^p := \left\{ x = (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

This is a Banach space with the norm $\|x\|_p := (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $x \in \ell^p$. Likewise, the space $\ell^\infty \subset \mathbb{R}^{\mathbb{N}}$ of bounded sequences is a Banach space with the supremum norm $\|x\|_\infty := \sup_{i \in \mathbb{N}} |x_i|$ for $x = (x_i)_{i \in \mathbb{N}} \in \ell^\infty$.

(iii) Let $(M, \mathcal{A}, \mu)$ be a measure space, i.e. M is a set, $\mathcal{A} \subset 2^M$ is a σ-algebra, and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. Fix a constant $1 \leq p < \infty$. A measurable function $f : M \to \mathbb{R}$ is called **p-integrable** if $\int_M |f|^p \, d\mu < \infty$ and the space of p-integrable functions on M will be denoted by

$$\mathcal{L}^p(\mu) := \left\{ f : M \to \mathbb{R} \,\middle|\, f \text{ is measurable and } \int_M |f|^p \, d\mu < \infty \right\}.$$

The function $\mathcal{L}^p(\mu) \to \mathbb{R} : f \mapsto \|f\|_p$ defined by

$$\|f\|_p := \left(\int_M |f|^p \, d\mu \right)^{1/p}$$

is nonnegative and satisfies the triangle inequality (Minkowski's inequality). However, in general it is not a norm, because $\|f\|_p = 0$ if and only if f vanishes **almost everywhere** (i.e. on the complement of a set of measure zero). To obtain a normed vector space, one considers the quotient

$$L^p(\mu) := \mathcal{L}^p(\mu)/\sim,$$

where

$$f \sim g \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad f = g \text{ almost everywhere.}$$

The function $f \mapsto \|f\|_p$ descends to this quotient space and, with this norm, $L^p(\mu)$ is a Banach space In this example it is often convenient to abuse notation and use the same letter f to denote a function in $\mathcal{L}^p(\mu)$ and its equivalence class in the quotient space $L^p(\mu)$.

(iv) Let $(M, \mathcal{A}, \mu)$ be a measure space, denote by $\mathcal{L}^\infty(\mu)$ the space of bounded measurable functions, and denote by

$$L^\infty(\mu) := \mathcal{L}^\infty(\mu)/\sim$$

the quotient space, where the equivalence relation is again defined by equality almost everywhere. Then the formula

$$\|f\|_\infty := \operatorname{ess\,sup}|f| = \inf\left\{c \geq 0 \,\middle|\, f \leq c \text{ almost everywhere}\right\}$$

defines a norm on $L^\infty(\mu)$, and $L^\infty(\mu)$ is a Banach space with this norm.

(v) Let M be a topological space. Then the space $C_b(M)$ of bounded continuous functions $f : M \to \mathbb{R}$ is a Banach space with the supremum norm

$$\|f\|_\infty := \sup_{p \in M}|f(p)|$$

for $f \in C_b(M)$.

(vi) Let $(M, \mathcal{A})$ be a measurable space, i.e. M is a set and $\mathcal{A} \subset 2^M$ is a σ-algebra. A **signed measure** on $(M, \mathcal{A})$ is a function $\mu : \mathcal{A} \to \mathbb{R}$ that satisfies $\mu(\emptyset) = 0$ and is σ-additive, i.e. $\mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ for every sequence of pairwise disjoint measurable sets $A_i \in \mathcal{A}$. The space $\mathcal{M}(M, \mathcal{A})$ of signed measures on $(M, \mathcal{A})$ is a Banach space with the norm given by

$$\|\mu\| := |\mu|(M) := \sup_{A \in \mathcal{A}}\big(\mu(A) - \mu(M \setminus A)\big).$$

for $\mu \in \mathcal{M}(M, \mathcal{A})$