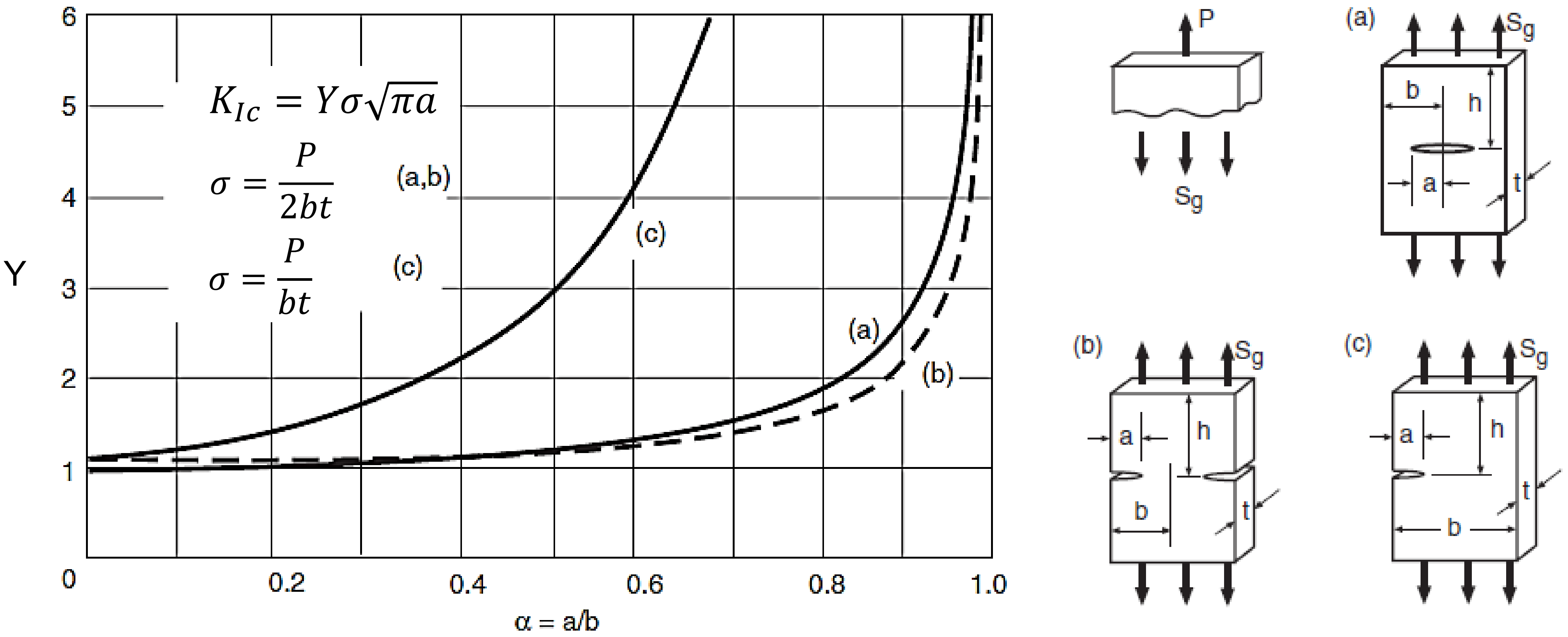
- The quantity ***Y*** is a dimensionless function that depends on the **geometry** and **loading configuration**, and usually also on the **ratio** of the **crack length** to another geometric dimension, such as the member **width** or **half-width**, *b*, as defined in the figure below.



Expressions for any $\alpha = a/b$

$$Y = 0.265(1-\alpha)^4 + \frac{0.857 + 0.265\alpha}{(1-\alpha)^{3/2}}$$

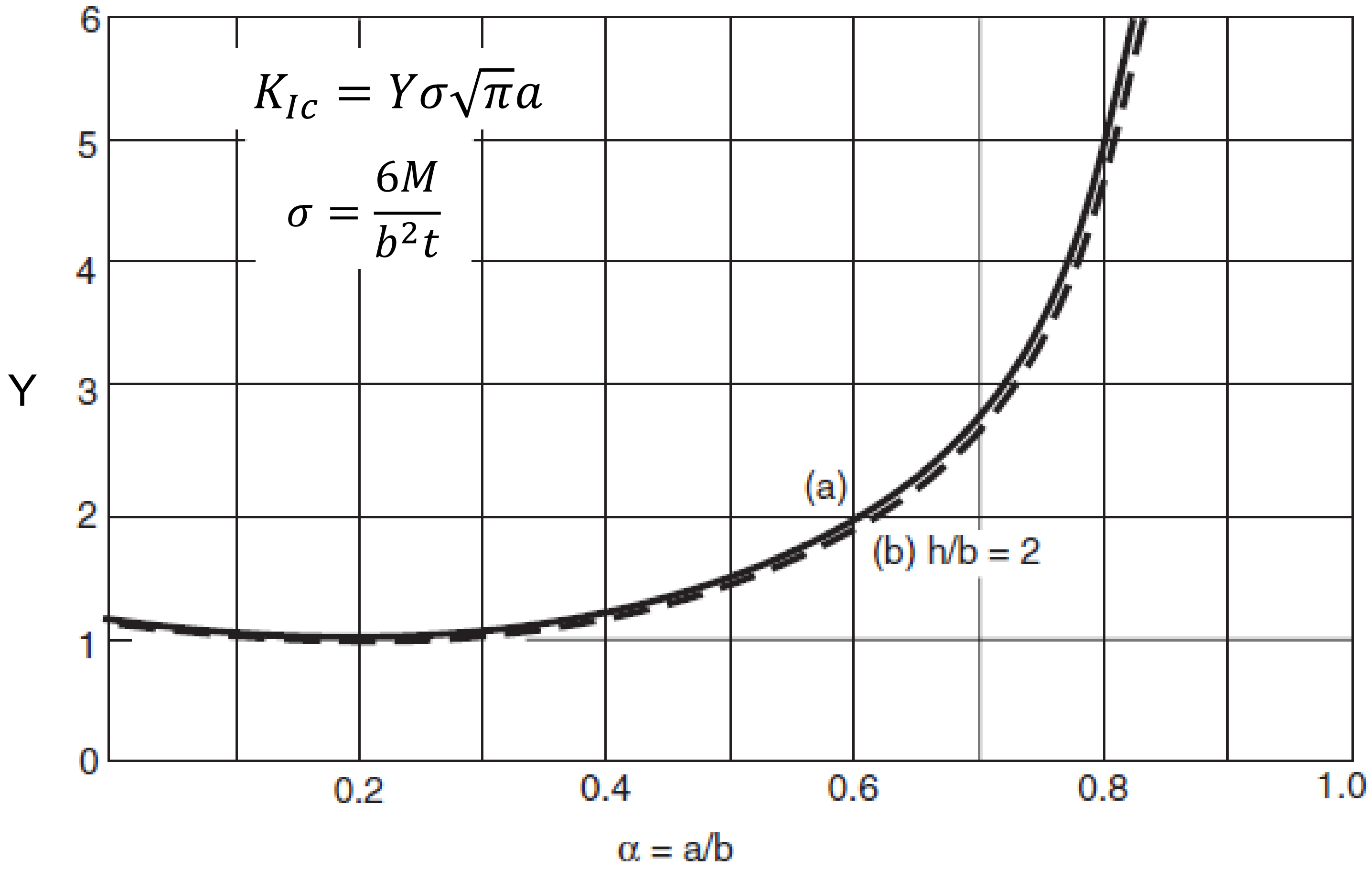
(h/b ≥1)

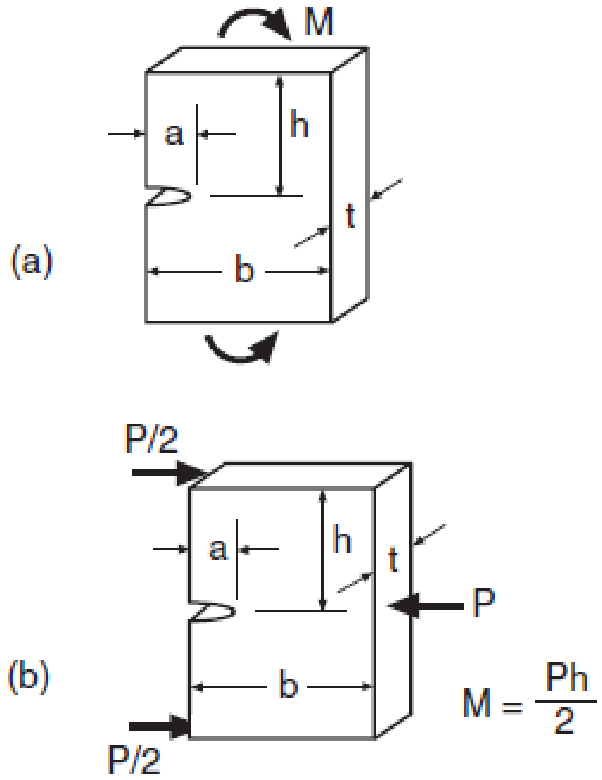
$$Y = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}}$$

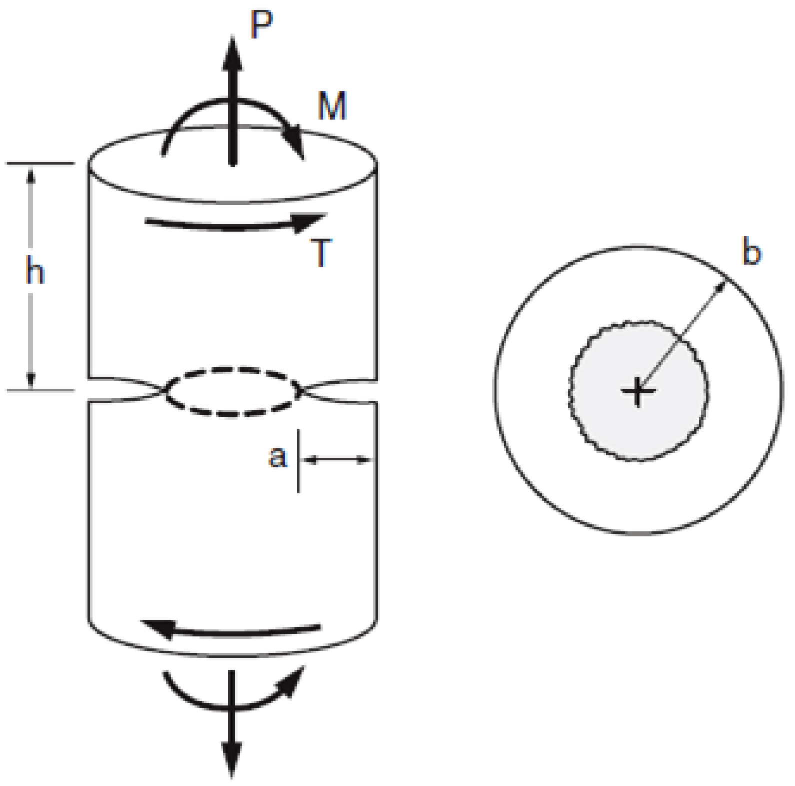
(h/b ≥1.5)

$$Y = (1 + 0.122\cos^4\frac{\pi\alpha}{2})\sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}}$$

(h/b ≥2)


$K_{Ic} = Y\sigma\sqrt{\pi a}$
$\sigma = \frac{6M}{b^2 t}$
Y
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1.0
α = a/b
(a)
(b) h/b = 2


M
a
h
t
(a)
b
P/2
a
h
t
P
(b)
b
M = Ph/2
P/2



$K_{Ic} = Y\sigma\sqrt{\pi a}$

$\alpha = a/b$

$\beta = 1 - \alpha$

(a) Axial load P: $\sigma = \dfrac{P}{\pi b^2}$, $F = 1.12$ $(10\%,\ a/b \le 0.21)$

$$F = \frac{1}{2\beta^{1.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 - 0.363\beta^3 + 0.731\beta^4\right]$$

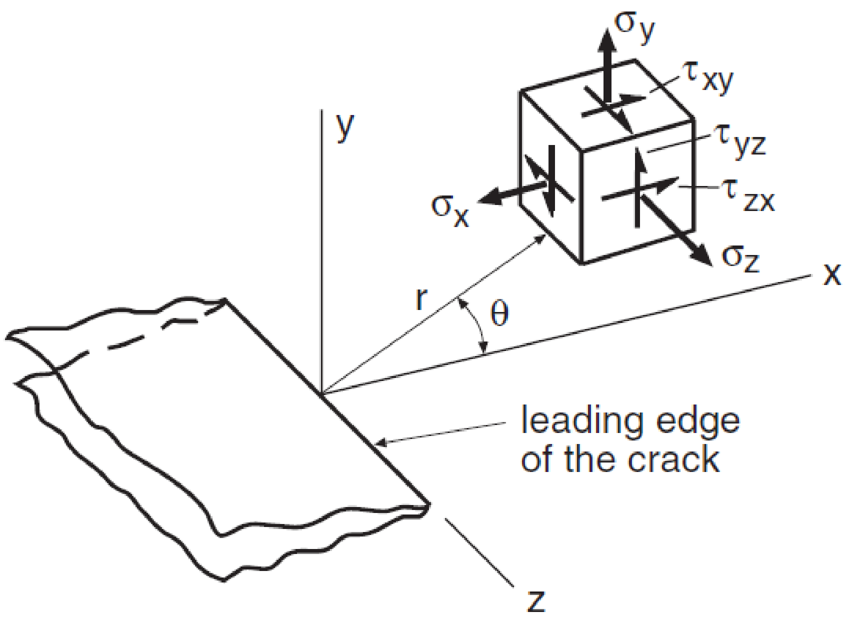
(b) Bending moment M: $\sigma = \dfrac{4M}{\pi b^3}$, $F = 1.12$ $(10\%,\ a/b \le 0.12)$

$$F = \frac{3}{8\beta^{2.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 + \frac{5}{16}\beta^3 + \frac{35}{128}\beta^4 + 0.537\beta^5\right]$$

(c) Torsion T, $K = K_{III}$: $\sigma = \dfrac{2T}{\pi b^3}$, $F = 1.00$ $(10\%,\ a/b \le 0.09)$

$$F = \frac{3}{8\beta^{2.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 + \frac{5}{16}\beta^3 + \frac{35}{128}\beta^4 + 0.208\beta^5\right]$$

- For **any case** of Mode I loading, the **stresses** near the crack tip depend on r and θ as follows:



$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \cdots \qquad \text{(a)}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] + \cdots \qquad \text{(b)}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + \cdots \qquad \text{(c)}$$

$$\sigma_z = 0 \qquad \text{(plane stress)} \qquad \text{(d)}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right) \qquad (\text{plane strain; } \varepsilon_z = 0) \qquad \text{(e)}$$

$$\tau_{yz} = \tau_{zx} = 0 \qquad \text{(f)}$$

Safety factor

- Where cracks may be present, safety factors against yielding, need to be **supplemented** by safety factors against brittle **fracture.**

- The safety factor on K, and thus on σ, is

$$X_K = \frac{K_{Ic}}{K} = \frac{K_{Ic}}{Y\sigma\sqrt{\pi a}}$$

- The value of a_c is available from;

$$K_{Ic} = Y_c\sigma\sqrt{\pi a_c}$$

- Combining the previous two equations leads to the following safety factor on crack length:

$$X_a = \frac{a_c}{a} = \left(\frac{Y}{Y_c}\, X_K\right)^2$$

Plastic-Zone Size Correction

- When yielding occurs at the crack tip, it becomes **blunted**; that is, the crack surfaces separate **without** any crack extension
- the plastic zone (radius r_y) will then be **embedded in an elastic stress** field.
- When the whole of the **reduced** section yields, the plastic zone spreads to the edges of the sample, and K **does not have any validity.** For **small** plastic zone in relation to the crack length;

$$\sigma_y = \frac{K}{\sqrt{2\pi r_y}} \Rightarrow r_y = \frac{1}{2\pi}\left(\frac{K}{\sigma_y}\right)^2$$

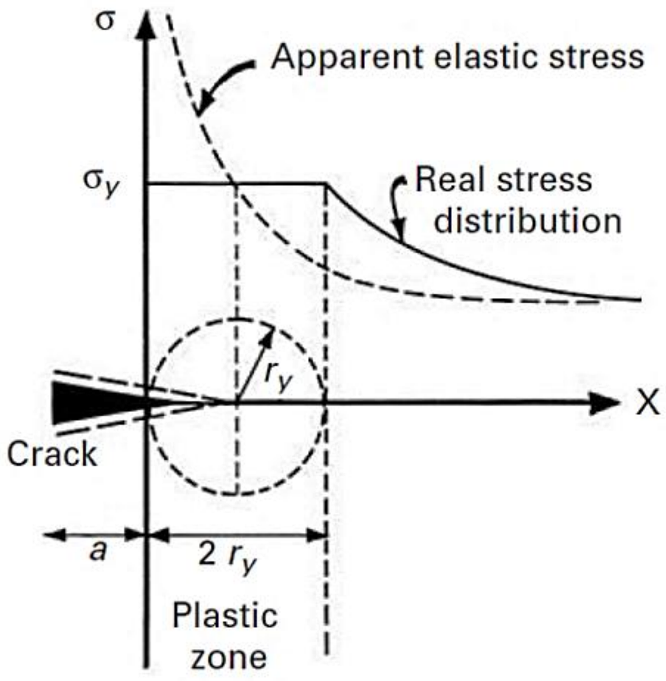
For plane stress

Irwin, taking into account the plastic constraint factor in the case of plane strain, gave

$$r_y \approx \frac{1}{6\pi}\left(\frac{K}{\sigma_y}\right)^2$$

For plane strain

- The effective crack length is, then, $(2a)_{eff} = 2(a + r_y)$



Crack Opening Displacement

- The development of a plastic zone at the tip of the crack results in a displacement of the faces (called **Crack Opening Displacement** COD) **without** crack extension.

- Consider a small crack in a brittle material. We have:

$$\sigma_c = K_{Ic}(\sqrt{\pi a})^{-1}, \qquad as\ a \to 0, \sigma_c \to \infty$$

But this **does not occur**. Instead, a plastic zone develops and may extend through the section

- In more **ductile** materials, the critical stress predicted by LEFM will be **higher** than σ_y. In the elastic case:

$$\text{COD} = \Delta = \frac{4\sigma}{E}\sqrt{(a^2 - x^2)}$$

At the center of the crack (x = 0), the maximum opening is $\Delta_{max} = \frac{4\sigma a}{E}$

- Applying the plastic zone correction, $\Delta = \frac{4\sigma}{E}\sqrt{\left(a + r_y\right)^2 - x^2}$

- The crack-tip opening displacement (CTOD), δ, is given for $x = a$ and $r_y \ll a$ as

$$\Delta = \frac{4\sigma}{E}\sqrt{2a_{eff}r_y}$$

Substituting the value of the plastic-zone radius, $r_y = \sigma^2 a/2\sigma_y^2$,

$$\delta = \frac{4K_I^2}{\pi E\sigma_y}$$

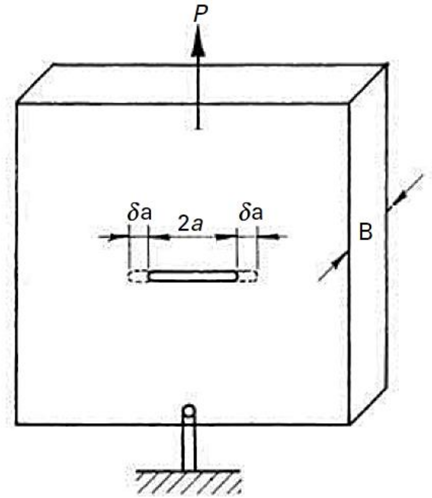
- The fracture occurs when $K_I = K_{Ic}$, which corresponds to $\delta = \delta_{Ic}$. Direct measurement of δ_c is not easy. An indirect way is the following. We have

$$\Delta = \frac{4\sigma}{E}\sqrt{(a + r_y)^2 - x^2} \quad \Rightarrow \quad \Delta = \frac{4\sigma}{E}\sqrt{a^2 + 2ar_y + r_y^2 - x^2}$$

$$\therefore \Delta = \frac{4\sigma}{E}\left(a^2 - x^2 + \frac{E^2}{16\sigma^2}\delta^2\right)^{1/2}$$

Crack Extension Force G

- The concept of the crack extension force G, attributed to Irwin, can be interpreted as a **generalized force**.
- Before the crack extension, the potential energy stored in body shown here was

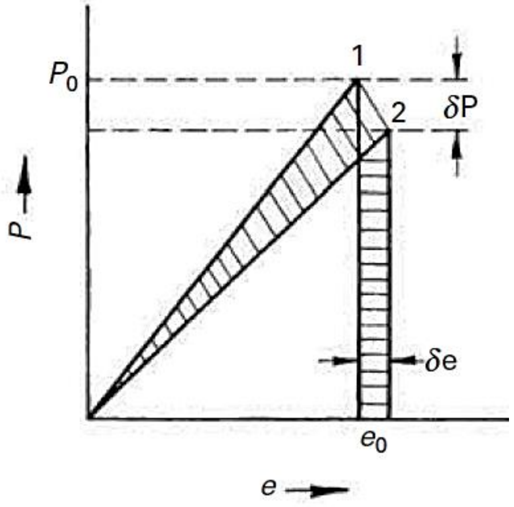


$$U_1 = \frac{1}{2} P e$$

After the crack extension,

$$U_2 = \frac{1}{2}(P - \delta P)(e + \delta e)$$

- We can write an equation for G, the crack extension force per unit length, as



$$G.t.\delta a = U_2 - U_1 = \delta U$$

The change in elastic strain energy with respect to the crack area, in the limit of the area going to zero, equals the crack extension force.

$$G = \lim_{\delta A \to 0} \frac{\delta U}{\delta A}$$ where $\delta A = t\,\delta a$.

- It is convenient to evaluate G in terms of the compliance c of the sample, defined as $e=cP$

$$\therefore \delta U = U_2 - U_1 = \frac{1}{2}(P - \delta P)(e + \delta e) - \frac{1}{2}Pe$$

or

$$\delta U = \frac{1}{2}P\delta e - \frac{1}{2}e\delta P - \frac{1}{2}\delta P\delta e$$

however;

$$\delta e = c\delta P + P\delta c$$

$$\delta U = \frac{1}{2}Pc\delta P + \frac{1}{2}P^2\delta c - \frac{1}{2}e\delta P - \frac{1}{2}e(\delta P)^2 - \frac{1}{2}P\delta P\delta c$$

$$\therefore \delta U = \frac{1}{2}Pc\delta P + \frac{1}{2}P^2\delta c - \frac{1}{2}Pc\delta P \quad \Rightarrow \quad \delta U = \frac{1}{2}P^2\delta c$$

$$G = \lim_{\delta A \to 0}\frac{\delta U}{\delta A} = \lim_{\delta A \to 0}\frac{\frac{1}{2}P^2\delta c}{\delta A}$$

or

$$G = \frac{1}{2}\frac{P^2\delta c}{t\delta a}$$

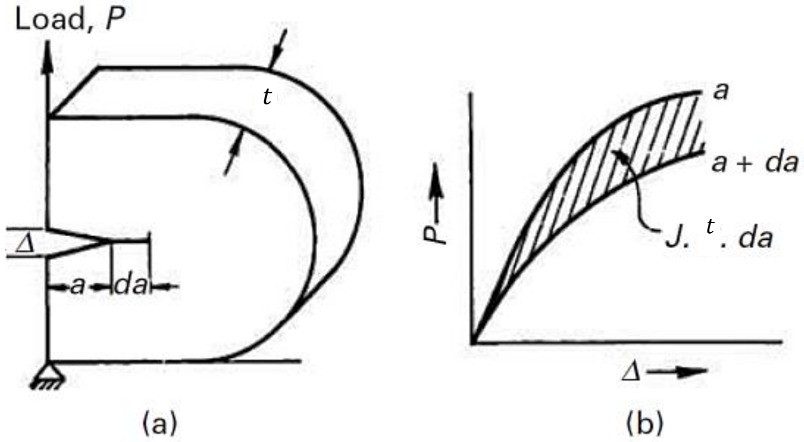
J Integral

- It provides a value of **energy** required to **propagate a crack** in an elastic-plastic material.
- On the basis of the theory of conservation of energy, Eshelby showed that the integral J is **equal to zero** for a closed contour; that is,

$$J = \int \left(W dx_2 - T \frac{\partial u}{\partial x_1} ds \right) = 0$$

where: $W = \int_0^{\Sigma_{ij}} \sigma_{ij} d\varepsilon_{ij}$ the strain energy per unit volume

T is the tension vector (traction), ds is an element of length along the contour, and u is the displacement in the x_1 direction.

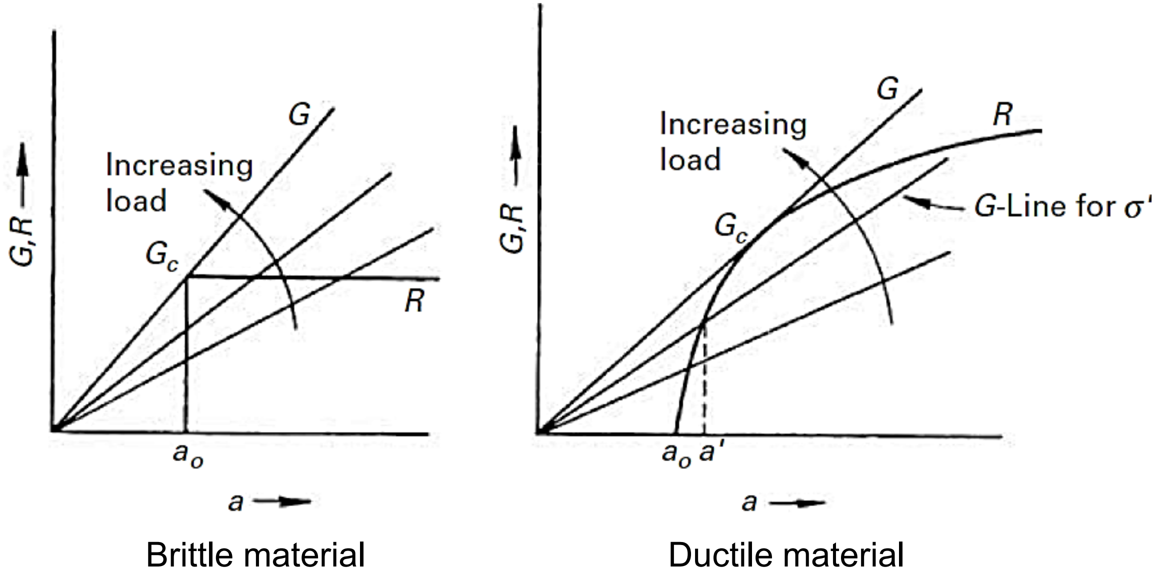


- the J integral represents the difference in the potential energies of identical bodies containing cracks of length a and $a + da$. For a body of thickness t, this can be written as

$$J = \frac{1}{t} \frac{\partial U}{\partial a}$$

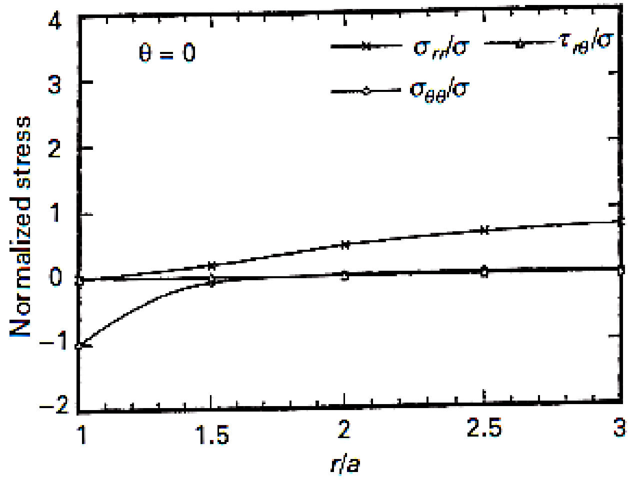
R Curve

- The R curve characterizes the **resistance** of a material to fracture (as a **function of crack growth**) during **slow and stable** propagation of a crack.
- According to Irwin, failure will **occur** when the rate of change of the crack extension force ($\partial G/\partial a$) **equals** the rate of change of this resistance to crack growth in the material ($\partial R/\partial a$).
- The resistance of the material to crack growth, R, increases with an increase in the size of the plastic zone.
- Since the plastic zone size **increases nonlinearly** with a, R will also be expected to **increase nonlinearly** with a. G **increases linearly** with a.
- The R curve characterizes the **resistance** of a material to fracture (as a **function of crack growth**) during **slow and stable** propagation of a crack.
- The R curve for a brittle material is a "square" curve, and the crack does not extend at all until the contact is reached, at which point $G = G_c$ and the unstable fracture follows.



Example:

Determine the stresses at distances equal to 0, *a/*2, *a, 3a/2*, and *2a* from the surface of a spherical hole and for θ = 0 and π/2.



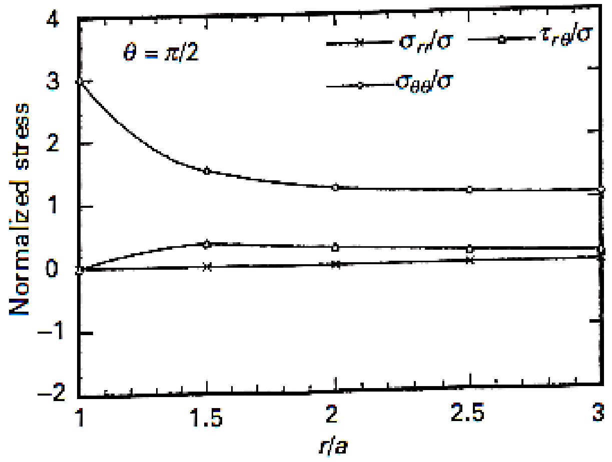
Solution:

By setting θ = 0, we have

$$\sigma_{rr}=\frac{\sigma}{2}\left(2-\frac{5a^2}{r^2}+\frac{3a^4}{r^4}\right)\qquad \sigma_{\theta\theta}=\frac{\sigma}{2}\left(\frac{a^2}{r^2}-\frac{3a^4}{r^4}\right)\qquad \tau_{r\theta}=0$$

For θ = 0

$$\sigma_{rr}=\frac{\sigma}{2}\left(\frac{3a^2}{r^2}-\frac{3a^4}{r^4}\right)\qquad \sigma_{\theta\theta}=\frac{\sigma}{2}\left(2+\frac{a^2}{r^2}+\frac{3a^4}{r^4}\right)\qquad \tau_{r\theta}=0$$



We calculate the stresses for *r = 0, a, 3a/2*, and *2a* and plot them as shown in the figure below in terms of a dimensionless parameter r/a.

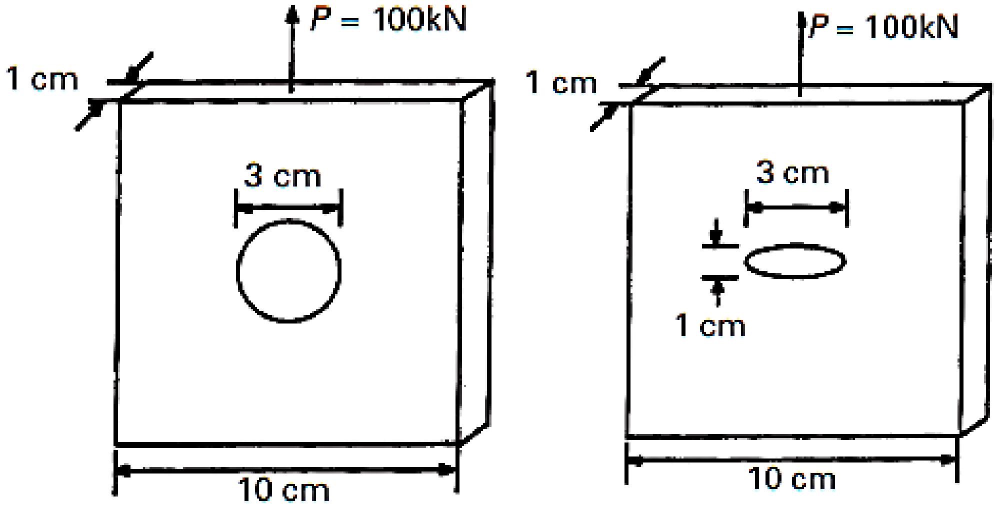
Example:

Two flat plates are being pulled in tension. (See the figure below) The flow stress of the materials is 150 MPa.

(a) Calculate the maximum stresses in the plate.

(b) Will the material flow plastically?

(c) For which configuration is the stress higher?



Solution:

(a) By setting $\theta = 0$, we have

$$\sigma = \frac{P}{A} = \frac{100\ \text{kN}}{10\ \text{cm}\ \times 1\ \text{cm}} = 100\ \text{MPa}$$

$$\sigma_{max} = \sigma\left(1 + 2\frac{a}{b}\right)$$

Circular hole ⇨ $a = b = 3/2\ \text{cm} = 1.5\ cm$ ⇨ $\sigma_{max} = 100 \times \left(1 + 2 \times \frac{1.5}{1.5}\right) = 700\ MPa$

(b) Yes, because in both cases, the stress is greater than the flow stress (150 MPa).

(c) The elliptical hole has higher stress than the circular one.

Superposition for Combined Loading

- **Stress intensity** solutions for **combined** loading can be obtained by **superposition**—that is, by **adding** the contributions to K from the individual load components.
- Consider an **eccentric load** applied a distance *e* from the centerline of a member with a single edge crack, as shown in the figure.
- The contribution to K from the centrally applied tension may be determined from:

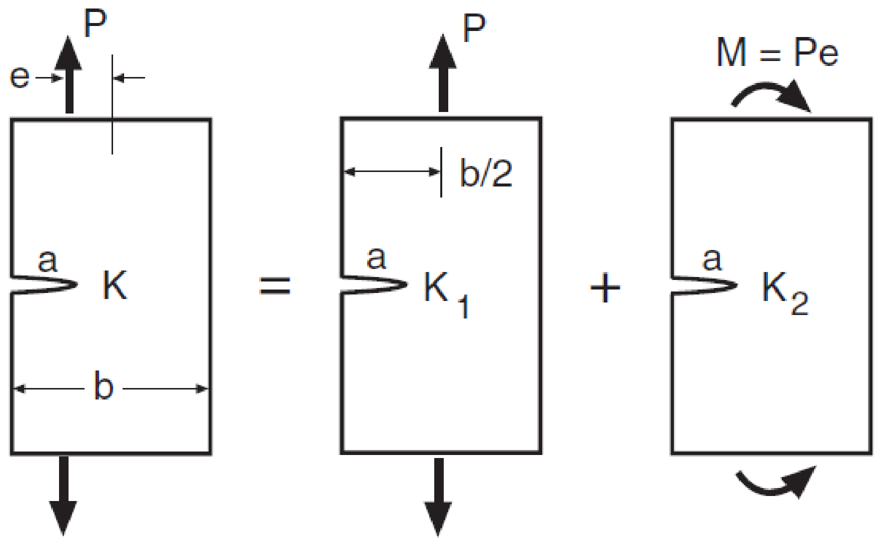
$$K_1 = Y_1 S_1 \sqrt{\pi a}$$

$$S_1 = \frac{P}{bt}$$

- The contribution from bending may be determined from

$$K_2 = Y_2 S_2 \sqrt{\pi a}$$

$$S_2 = \frac{6M}{b^2 t} = \frac{6Pe}{b^2 t}$$



- Hence, the **total stress intensity** due to the eccentric load is obtained by **summing** the two solutions and using substitutions from the previous equations; it is

$$K = K_1 + K_2 = \frac{P}{bt}(Y_1 + \frac{6Y_2 e}{b})\sqrt{\pi a}$$

where the particular a/b that applies is used to separately determine Y_1 and Y_2 for tension and bending, respectively.

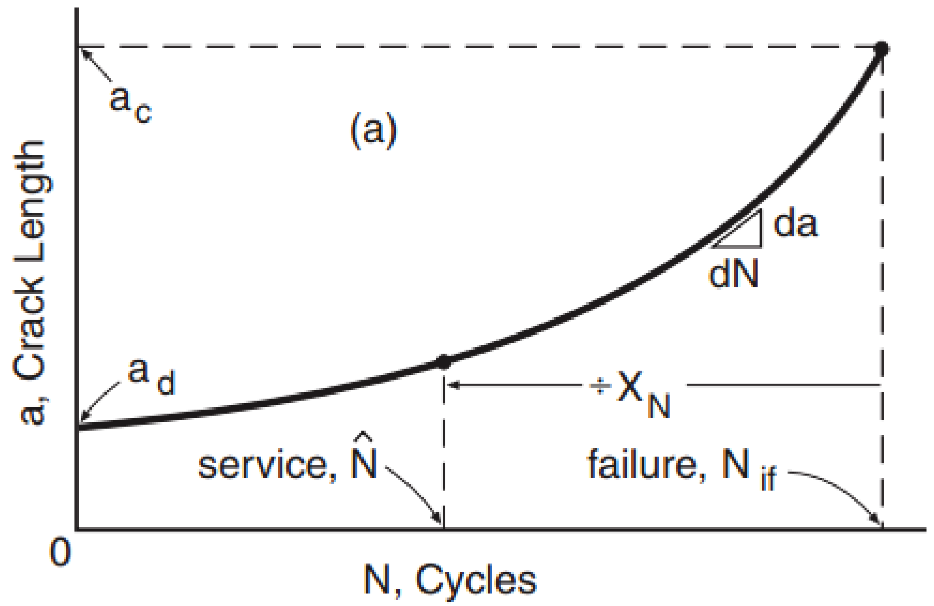
Definitions for Fatigue Crack Growth

- Consider a growing crack that increases its length by an amount Δa due to the application of a number of cycles ΔN.
- A value of fatigue crack growth rate, $\mathrm{d}a/\mathrm{d}N$, is the slope at a point on an a versus N curve as shown in the figure.
- For fatigue crack growth work, it is conventional to use the **stress range** ΔS and the **stress ratio** R,

$$\Delta S = S_{max} - S_{min}, \qquad R = \frac{S_{min}}{S_{max}}$$

- The primary variable affecting the growth rate of a crack is the **range** of the stress intensity factor. This is calculated from the stress range ΔS:

$$\Delta K = Y\,\Delta S\sqrt{\pi a}$$



- The maximum, minimum, range, and R-ratio for K during a loading cycle are respectively given by:

$$K_{max} = Y\,S_{max}\sqrt{\pi a} \qquad K_{min} = Y\,S_{min}\sqrt{\pi a}$$

$$\Delta K = K_{max} - K_{min}, \qquad R = \frac{K_{min}}{K_{max}}$$

- Also, it may be convenient, especially for laboratory test specimens, to use the alternative expression of K in terms of applied force P,

$$\Delta K = Y_P \frac{\Delta P}{t\sqrt{b}} \qquad R = \frac{P_{min}}{P_{max}}$$

Describing Fatigue Crack Growth Behaviour of Materials

- For a given material and set of test conditions, the crack growth behaviour can be described by the relationship between **cyclic crack growth rate da/dN and stress intensity range K.**

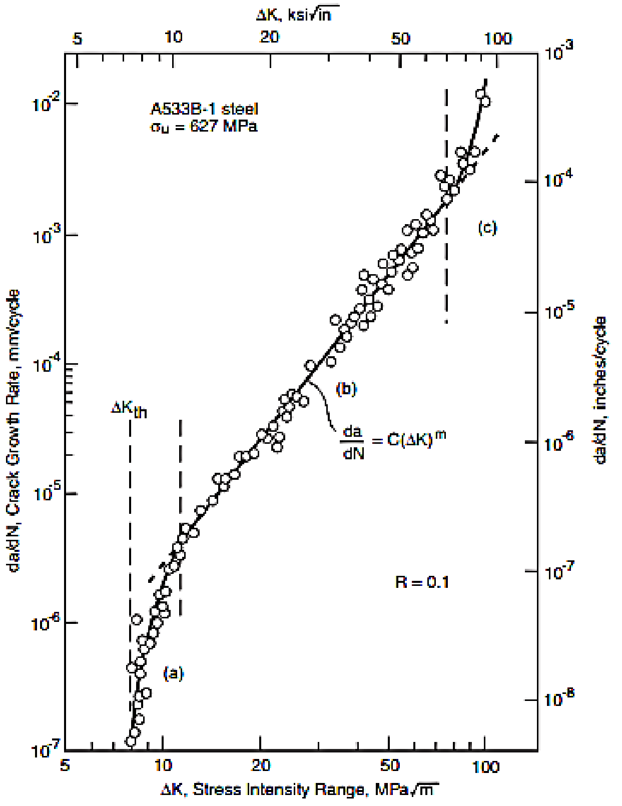
- At intermediate values of K, there is often a straight line on the log–log plot, as in this case. A relationship representing this line is

$$\frac{da}{dN} = C(\Delta K)^m$$

This equation is identified with **Paul Paris** in early 1960.

- At **low growth rates**, the curve generally becomes **steep** and appears to approach a vertical asymptote denoted K_{th}, which is called the **fatigue crack growth threshold**.

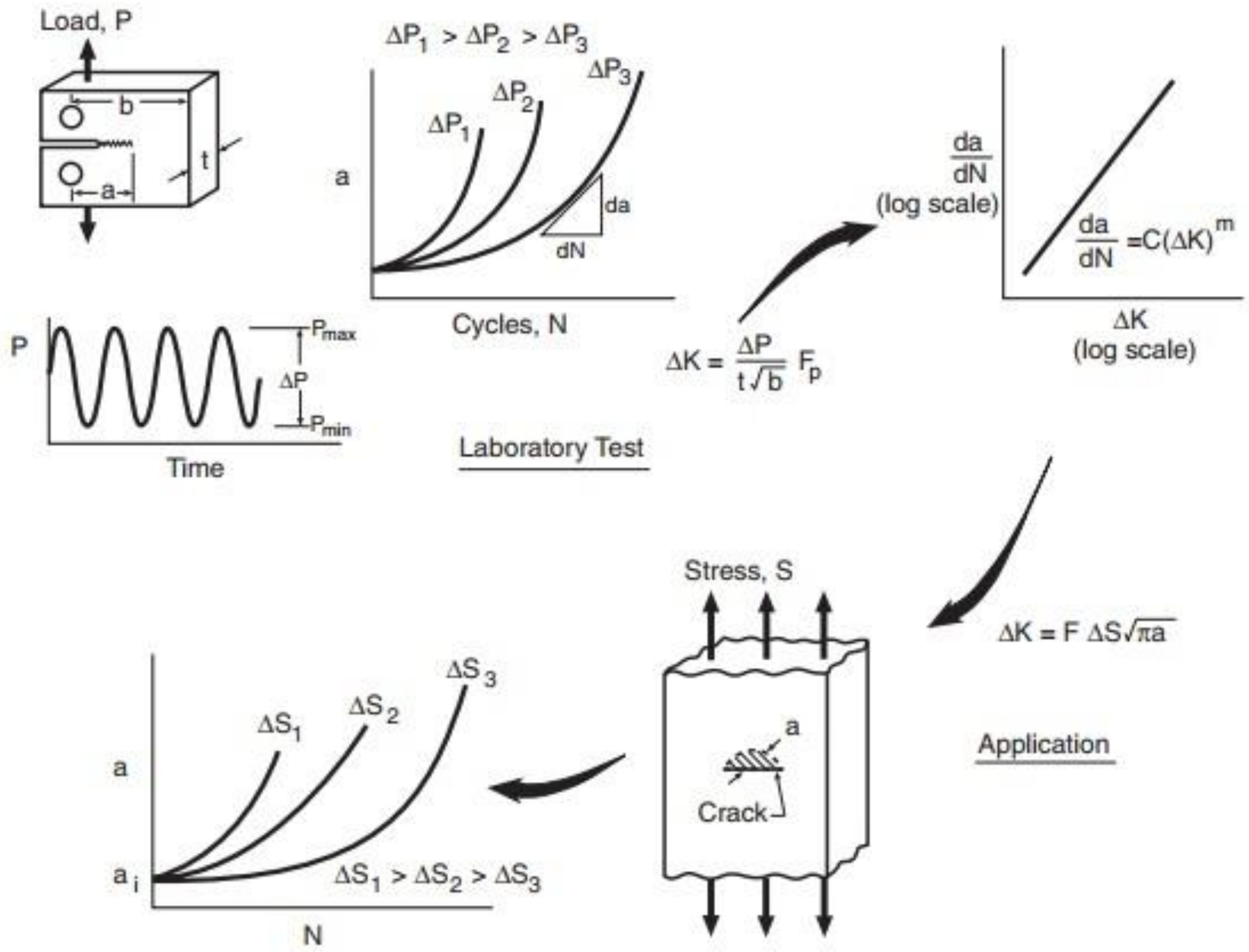
- This quantity is interpreted as a **lower limiting** value of K below which crack growth **does not** ordinarily occur.



- At **high growth rates**, the curve may again become **steep**, due to **rapid unstable** crack growth just prior to final failure of the test specimen.

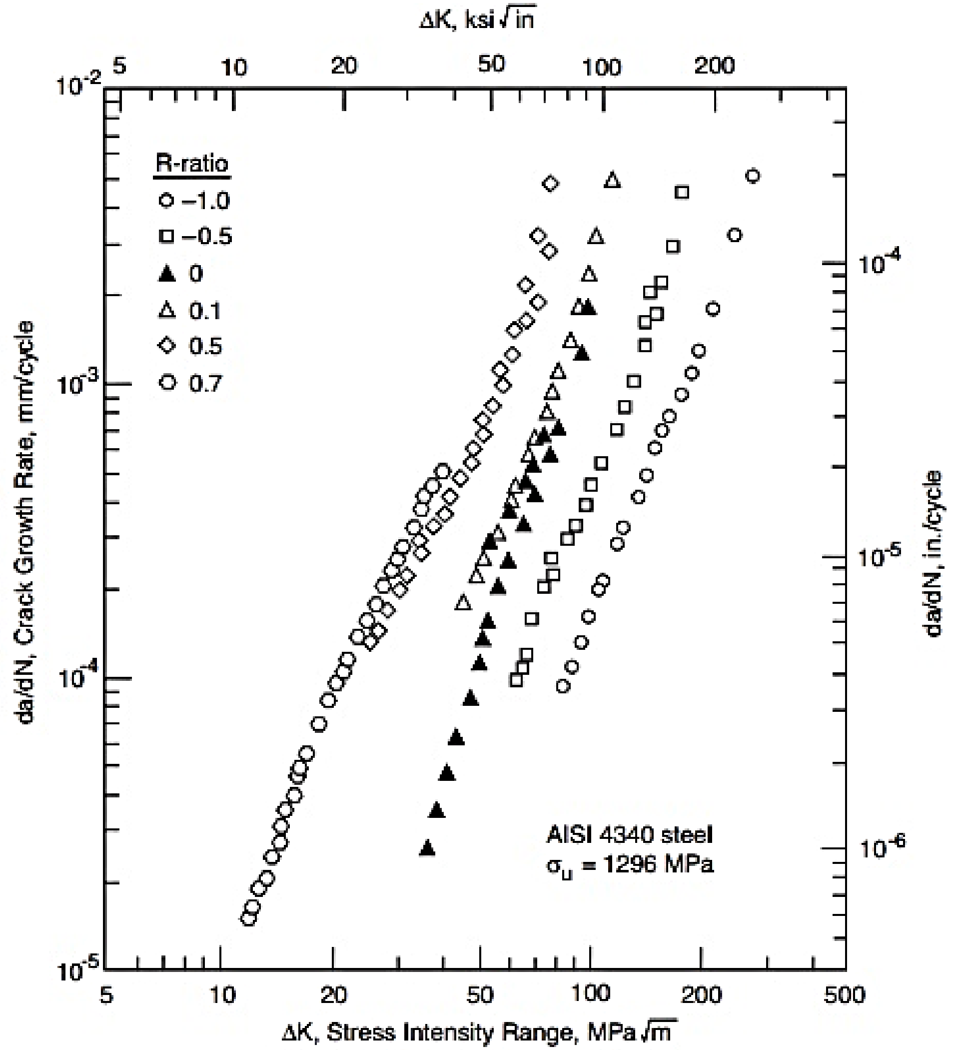
- **Rapid unstable growth** at high K sometimes involves **fully** plastic yielding. In such cases, the use of K for this portion of the curve is **improper**, as the theoretical limitations of the K concept are exceeded.

- The logical path involved in evaluating the crack growth behaviour of a material and using the information is summarized in the figure.



Example:

Obtain approximate values of constants *C* and *m*, and give the equation of **Paul Paris** for the data at *R* = 0.1 in the given figure.



Solution:

These data appear to fall along a straight line on this log–log plot, so it is reasonable to apply Paris equation. Aligning a straight edge with the data gives a line that passes near two points as follows:

$$\Delta K, \frac{da}{dN} = (21, 10^{-5}) \text{ and } (155, 10^{-2}))$$

$$da/dN_A = C(\Delta K_A)^m \qquad da/dN_B = C(\Delta K_B)^m$$

$$\frac{da/dN_A}{da/dN_B} = \left(\frac{\Delta K_A}{\Delta K_B}\right)^m$$

$$m = \frac{\log(da/dN_A) - \log(da/dN_B)}{\log \Delta K_A - \log \Delta K_B}$$

$$m = \frac{\log 10^{-5} - \log 10^{-2}}{\log 21 - \log 155} = 3.456$$

Next, obtain C by substituting this m and either known point into Paris equation:

$$10^{-5}\,\frac{mm}{cycle} = C(21\ MPa\sqrt{m})^{3.456} \quad \Rightarrow \quad C = 2.696 \times 10^{-10}\ \frac{mm/cycle}{(MPa\sqrt{m})^{m}}$$

Example:

A thin cylinder has a diameter of 1.5 m and a wall thickness of 100 mm. the working internal pressure of the cylinder is 24 MN/m^2 and KIC for the material is 54 $MN/m^{3/2}$. Estimate the size of the largest flaw that the cylinder can contain. Non-destructive testing by ultrasonic methods reveals that the cylinder contain cracks up to 2a=0.2 mm in length. Laboratory tests show that the crack-growth rate under cyclic loading is given below. Estimate the number of pressurization cycles that the cylinder can safely withstand.

$$\frac{da}{dN} = 3x10^{-12} (\Delta K)^{3.8}$$

Solution

1. Calculate σ
2. K= K_{IC} to find a
3. $\frac{da}{dN} = C(\Delta K)^m$ where $\Delta K = K_{max}$ to find N