Fatigue Strength

- In **some design applications** the number of load cycles the element is subjected to is limited (less than 10^6) and therefore there is **no need to design for infinite life** using the endurance limit.

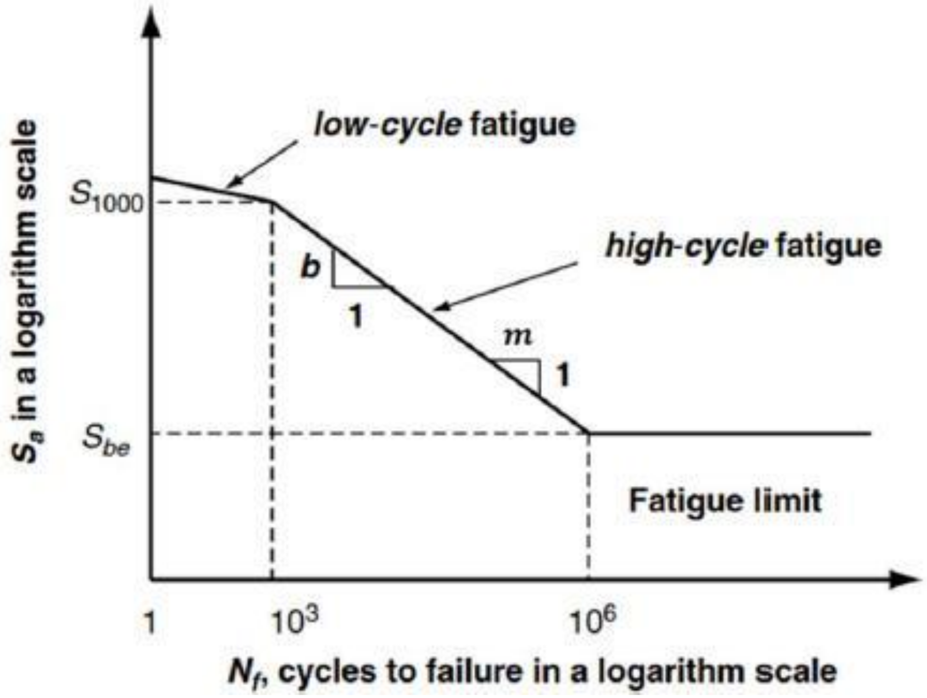
- In such cases we need to find the **fatigue strength** associated with the desired life.

- For the high cycle fatigue (10^3 to 10^6), the line equation is $S_f = a.N^b$ where the constant "a" (y intercept) and "b" (slope) are determined from the end point $(S_f)_{10^3}$ and $(S_f)_{10^6}$ as:

$$a = \frac{(S_f)^2_{10^3}}{S_e} \quad and \quad b = -\frac{\log(\sigma'_f/S_e)}{\log(2N_e)}$$

where σ'_f is the true stress at fracture and for steels with $H_B \leq 500$, it is approximated as:

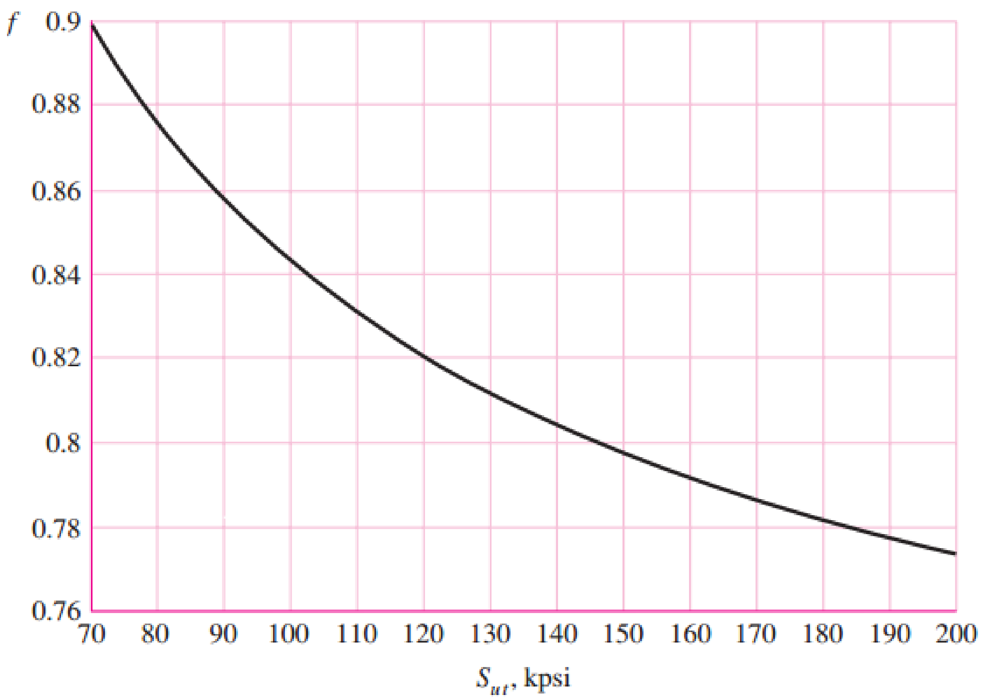
$$\sigma'_f = S_{ut} + 345 \text{ MPa}$$



- $(S_f)_{10^3}$ can be related to S_{ut} as:

$$(S_f)_{10^3} = f.S_{ut}$$

where f can be obtained from the figure here



- If the value of f is known, the constant b directly found as:

$$b = -\frac{1}{3} log\left(\frac{f.S_{ut}}{S_e}\right)$$

and a can be rewritten as:

$$a = \frac{(f.S_{ut})^2}{S_e}$$

Thus for $10^3 \leq N \leq 10^6$, the fatigue strength associated with a given life (N) is:

$$\left(S_f\right)_N = a.N^b$$

and the fatigue life (N) at a given fatigue:

$$N = \left(\frac{\sigma}{a}\right)^{1/2}$$

- Studies show that for **ductile materials**, the fatigue stress concentration factor (k_f) reduces for $N<10^6$, however the **conservative** approached is to use k_f as is.

Example:

For a rotating beam specimen made of 1050 CD steel, find;

1. The endurance limit (N=10^6)
2. The fatigue strength corresponding to ($5x10^4$) cycles to failure
3. The expected life under a completely reversed stress of 400 MPa

Solution:

From *Table A-20* $S_{ut} = 690\ MPa$

a) $S_e{}' = 0.5(S_{ut}) = 345\ MPa$

Note that no modifications are needed since it is a specimen: $S_e = S_e{}'$

b) $\sigma_f' = S_{ut} + 345 = 1035\ MPa$

$$b = -\frac{log\,(\sigma_f'/S_e)}{log(2N_e)} = -\frac{log(1035/345)}{log(2\times 10^6)} = -0.0757$$

$$f = \frac{\sigma_f'}{S_{ut}}(2\times 10^3)^b = \frac{1035}{690}(2\times 10^3)^{-0.0757} = 0.844$$

$$a = \frac{(f\,S_{ut})^2}{S_e} = \frac{(0.844\times 690)^2}{345} = 982.4\ MPa$$

OR, easier, from Figure 6-18: $f \cong 0.845$

Then,

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{(0.845 \times 690)^2}{345} = 985.3\ MPa$$

$$b = -\frac{1}{3} log\left(\frac{f S_{ut}}{S_e}\right) = -\frac{1}{3} log\left(\frac{0.845 \ \times 690}{345}\right) = -0.076$$

$$(S_f)_N = aN^b \quad \rightarrow \quad (S_f)_{5\times10^4} = 982.4(5 \times 10^4)^{-0.0757}$$

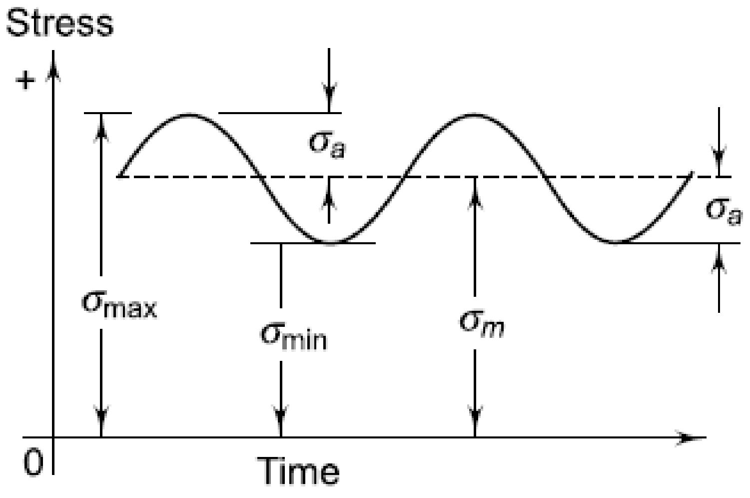
$$\rightarrow \quad (S_f)_{5\times10^4} = \boxed{433.1\ MPa}$$

c)

$$N = \left(\frac{\sigma}{a}\right)^{\frac{1}{b}} = \left(\frac{400}{982.4}\right)^{\frac{1}{-0.0757}} = \boxed{142.9 \times 10^3 cycles}$$

Characterizing Fluctuating Stress

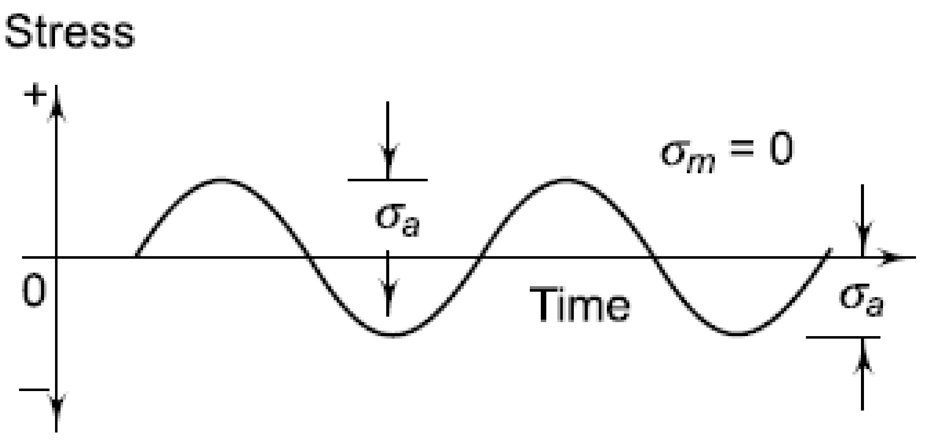
- In the rotating beam test, the specimen is subjected to **completely reversed stress** cycles ($\sigma_{max}=|\sigma_{min}|$).
- In the case of the **rotating shaft** subjected to **both radial and axial loads** (such as with helical gears) the fluctuating stress pattern will be different since there will be a component of stress that is always present (due to the axial load).
- The following stress components can be defined for distinguishing different states of fluctuating stress:
  - ✓ $\boldsymbol{\sigma_m}$: Mean or average stress $\quad \sigma_m = \frac{\sigma_{max}+\sigma_{min}}{2}$
  - ✓ $\boldsymbol{\sigma_r}$: Stress range $\quad \sigma_r = |\sigma_{max} - \sigma_{min}|$
  - ✓ $\boldsymbol{\sigma_a}$: Stress amplitude $\quad \sigma_a = \left|\frac{\sigma_{max}-\sigma_{min}}{2}\right|$



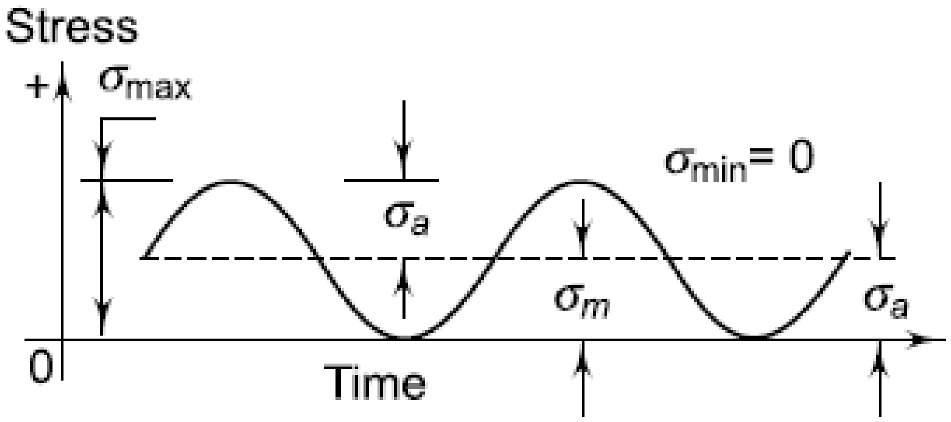
(a) Fluctuating stresses

- For uniform periodic fluctuating stress σ_m and σ_a are used to characterize the stress pattern.

- We also define:
  - ✓ stress ratio: $R = \frac{\sigma_{min}}{\sigma_{max}}$
  - ✓ Amplitude ratio: $A = \frac{\sigma_a}{\sigma_m}$
- Some common types of fluctuating stress:
  - ✓ **Completely reversed** stress $\sigma_m = 0$
    $\sigma_a = \sigma_{max} = |\sigma_{min}|$
  - ✓ **Repeated** stress
    Tension $\sigma_a = \sigma_m = \sigma_{max}/2$
    Compression $\sigma_m = \sigma_{min}/2$
  - ✓ **General** fluctuating stress (non-zero mean)
    $\sigma_a \neq \sigma_m \neq 0$



(c) Reversed stresses

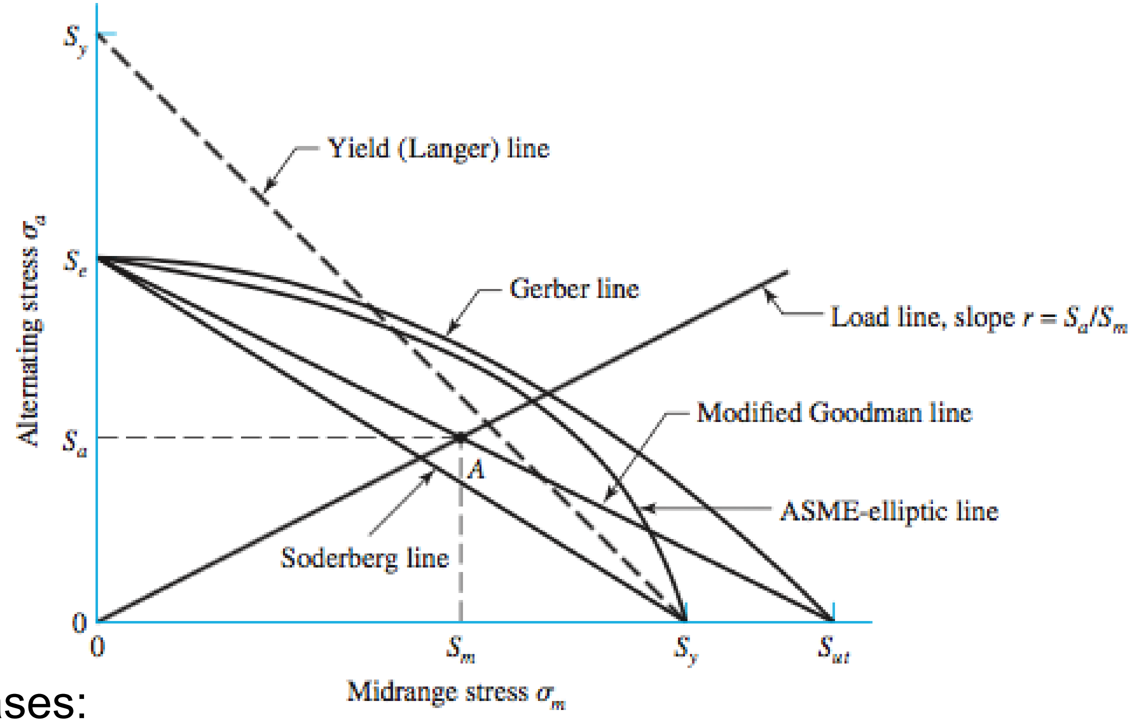


(b) Repeated stresses

Fatigue Failure Criteria for Fluctuating Stress

When a machine element is subjected to completely reversed stress (zero mean $\sigma_m = 0$) the endurance limit is obtained from the rotating beam test (after applying the necessary modification factors).
However, when the mean (midrange) stress is non-zero the situation is different and a **fatigue failure criteria** is needed.

- If we plot the alternating stress component (σ_a) vs. the mean stress component (σ_m), this will help in distinguishing the different fluctuating stress scenarios.
  - ✓ When $\sigma_m = 0$ & $\sigma_a \neq 0$, this will be a **completely reversed fluctuating stress**.
  - ✓ When $\sigma_m \neq 0$ & $\sigma_a = 0$, this will be a **static stress**.
  - ✓ Any combination of σ_m & σ_a will **fall between** the two extremes (completely reversed and static).



- Different theories are proposed to predict failure in such cases:

1. **Yield (Langer) line**: it connects S_y on the σ_a axis with S_y on the σ_m. But it is not realistic because S_y is usually larger than S_e.

$$S_a + S_m = S_y$$

2. **Soderberg line**: the most conservative, it connects S_e on σ_a axis with S_y on σ_m axis.

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n}$$

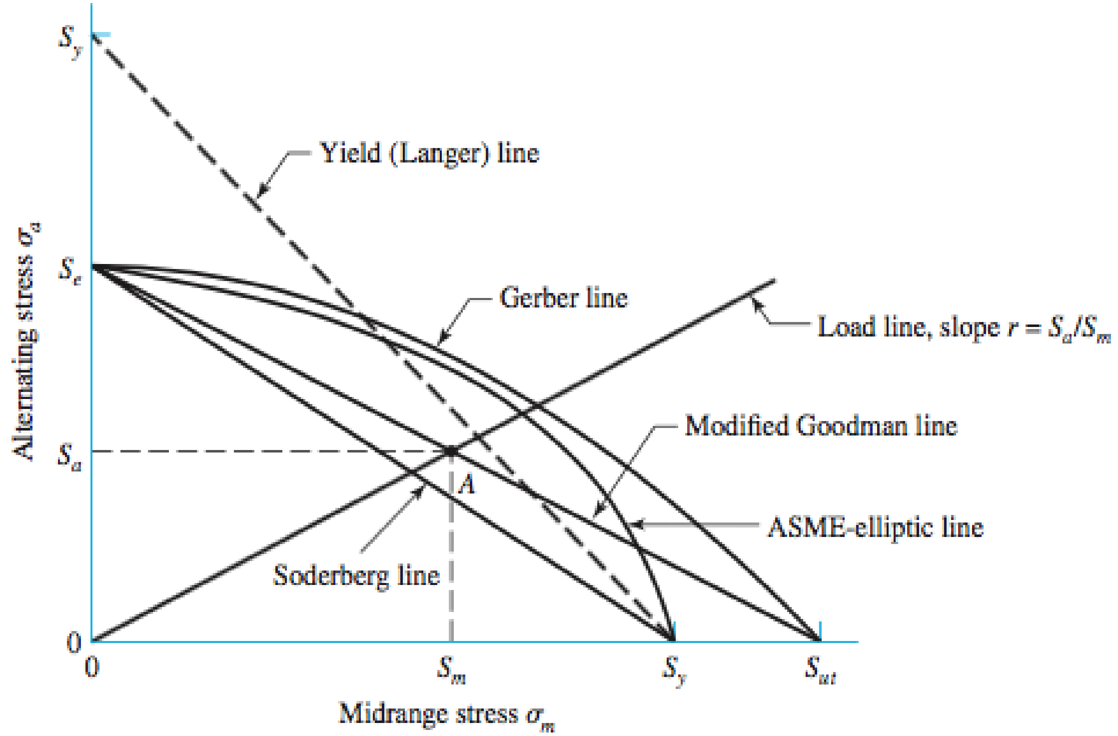
n : design factor

3. **ASME-elliptic line**: same as Soderberg but it uses an ellipse instead of the straight.

$$\left(\frac{n.\sigma_a}{S_e}\right)^2 + \left(\frac{n.\sigma_m}{S_y}\right)^2 = 1$$

4. **Goodman line**: it considers failure due to static loading to be at S_{ut} rather than S_y, thus it connects S_e on σ_a axis with S_{ut} on σ_m axis using a straight line.

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n}$$



5. **Gerber line**: same as Goodman but it uses a parabola instead of the straight line.

$$\frac{n.\sigma_a}{S_e} + \left(\frac{n.\sigma_m}{S_{ut}}\right)^2 = 1$$

- It should be noted that S_e is the **modified** endurance limit.
- The **fatigue stress concentration factor** (k_f) should be **multiplied with both** σ_a and σ_m for conservative results.
- The load line represents any combination of σ_a and σ_m, the intersection of the load line with any of the failure lines gives the limiting values S_a and S_m according to the line it intercepts.

Modified Goodman (Goodman and Langer)

- It combined the Goodman and Langer lines.
- The slope of the loading line passing through the intersection point of the two lines is called the **critical slope** and it is found as:

$$r_{critical} = S_a/S_m$$

where:

$$S_m = \frac{(S_y - S_e)S_{ut}}{S_{ut} - S_e} \quad \& \quad S_a = S_y - S_m$$

How?

- According to the **slope of the load line** ($r = \sigma_a/\sigma_m$), it could intersect any of the two lines:

$$r > r_{crit} \quad \rightarrow \quad S_a = \frac{r.S_e\, S_{ut}}{r.S_{ut} + S_e} \quad \& \quad S_m = \frac{S_a}{r}, \qquad n_f = \frac{S_a}{\sigma_a} = \frac{S_m}{\sigma_m} = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

$$r < r_{crit} \quad \rightarrow \quad S_a = \frac{r.S_y}{1+r} \quad \& \quad S_m = \frac{S_y}{1+r}, \qquad n_s = \frac{S_a}{\sigma_a} = \frac{S_m}{\sigma_m} = \frac{S_y}{\sigma_a + \sigma_m}$$

  - The second case here is considered to be a static yielding failure.

- The failure criteria are used in conjunction with a load line, r = S_a/S_m = σ_a/σ_m. Principal intersections are tabulated in the tables shown below. Formal expressions for fatigue factor of safety are given in the lower panel of the tables. The first row of each table corresponds to the fatigue criterion, the second row is the static Langer criterion, and the third row corresponds to the intersection of the static and fatigue criteria.

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for **Modified Goodman** and Langer Failure Criteria.

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r\,S_e S_{ut}}{r\,S_{ut} + S_e}$ $S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r\,S_y}{1+r}$ $S_m = \frac{S_y}{1+r}$
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ $\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = \frac{(S_y - S_e)\,S_{ut}}{S_{ut} - S_e}$ $S_a = S_y - S_m,\ r_{crit} = S_a/S_m$

Fatigue factor of safety

$$n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for **Gerber** and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r^2 S_{ut}^2}{2S_e}\left[-1 + \sqrt{1 + \left(\frac{2S_e}{r\,S_{ut}}\right)^2}\,\right]$ $S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r\,S_y}{1+r}$ $S_m = \frac{S_y}{1+r}$
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$ $\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = \frac{S_{ut}^2}{2S_e}\left[1 - \sqrt{1 + \left(\frac{2S_e}{S_{ut}}\right)^2\left(1 - \frac{S_y}{S_e}\right)}\,\right]$ $S_a = S_y - S_m,\ r_{crit} = S_a/S_m$

Fatigue factor of safety

$$n_f = \frac{1}{2}\left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e}\left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut}\sigma_a}\right)^2}\,\right] \qquad \sigma_m > 0$$

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for **ASME-elliptic** and Langer Failure Criteria.

Intersecting Equations	Intersection Coordinates
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$	$S_a = \sqrt{\frac{r^2 S_e^2 S_y^2}{S_e^2 + r^2 S_y^2}}$
Load line $r = S_a/S_m$	$S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = \frac{r S_y}{1 + r}$
Load line $r = S_a/S_m$	$S_m = \frac{S_y}{1 + r}$
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$	$S_a = 0, \ \frac{2 S_y S_e^2}{S_e^2 + S_y^2}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = S_y - S_a, \ r_{crit} = S_a/S_m$

Fatigue factor of safety

$$n_f = \sqrt{\frac{1}{(\sigma_a/S_e)^2 + (\sigma_m/S_y)^2}}$$

Combination of Loading Modes

- Calculate von Mises stresses for alternating and midrange stress states, σ'_a and σ'_m. When determining Se, do not use kc nor divide by k_f or k_{fs}. Apply k_f and/or k_{fs} directly to each specific alternating and midrange stress. If axial stress is present divide the alternating axial stress by k_c = 0.85.

$$\sigma'_a = \left\{ \left[(K_f)_{bending} (\sigma_a)_{bending} + (K_f)_{axial} \frac{(\sigma_a)_{axial}}{0.85} \right]^2 + 3 \left[(K_{fs})_{torsion} (\tau_a)_{torsion} \right]^2 \right\}^{1/2}$$

$$\sigma'_m = \left\{ \left[(K_f)_{bending} (\sigma_m)_{bending} + (K_f)_{axial} (\sigma_m)_{axial} \right]^2 + 3 \left[(K_{fs})_{torsion} (\tau_m)_{torsion} \right]^2 \right\}^{1/2}$$

- Apply these stresses to appropriate fatigue criteria shown previously above.
- Conservative check for localized yielding using von Mises stresses

$$\sigma'_a + \sigma'_m = \frac{S_y}{n}$$

Example:

A 40mm diameter bar has been machined from AISI 1045 CD bar. The bar will be subjected to a fluctuating tensile load varying from 0 to 100kN. Because of the ends fillet radius, k_f =1.85 is to be used. Find the critical mean and alternating stress values S_a and S_m and the fatigue factor of safety n_f according to the Modified Goodman fatigue criterion.

Solution:

From *Table A-20* $S_{ut} = 630\ MPa$ & $S_y = 530\ MPa$

$$S_e' = 0.5(S_{ut}) = 315\ MPa$$

Modifying factors:

- *Surface factor:* $k_a = 4.51(630)^{-0.265} = 0.817$ (*Table 6-2*)
- *Size factor:* $k_b = 1$ *since the loading is axial*
- *Loading factor:* $k_c = 0.85$ (*for axial loading*)
- *Other factors:* $k_d = k_e = k_f = 1$

➔ $S_e = k_a k_c S_e' = (0.817)(0.85)(315) = 218.8\ MPa$

Fluctuating stress: $\sigma = \frac{F}{A}$, $A = \frac{\pi}{4}d^2 = 1.257 \times 10^3\ mm^2$

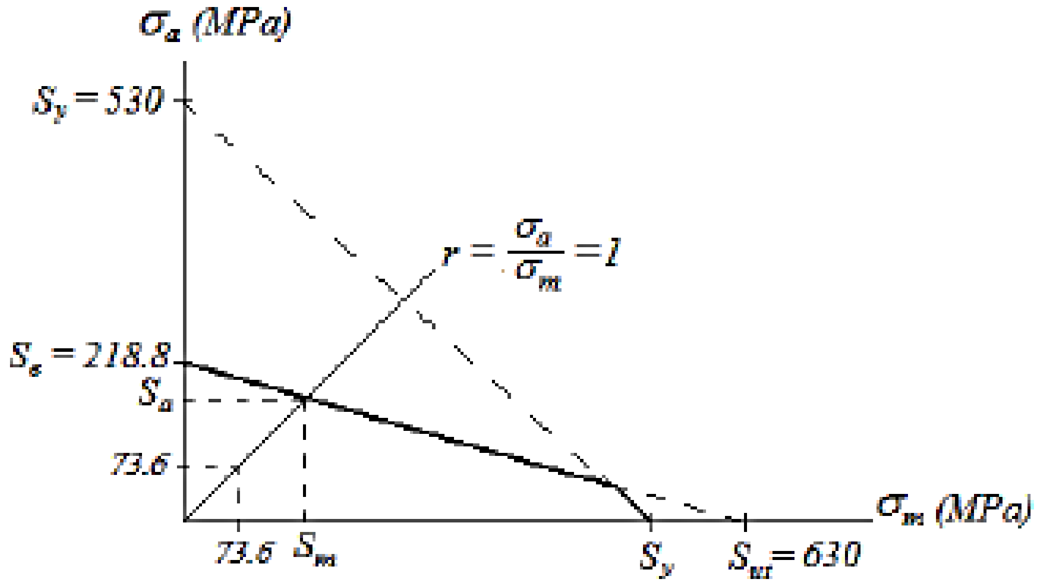
$$\sigma_{max} = \frac{100 \times 10^3}{1.257 \times 10^3} = 79.6\ MPa \quad \& \quad \sigma_{min} = 0$$

$$\sigma_{m_o} = \frac{\sigma_{max} + \sigma_{min}}{2} = 39.8\ MPa \quad \& \quad \sigma_{a_o} = \frac{\sigma_{max} - \sigma_{min}}{2} = 39.8\ MPa$$

Applying K_f to both components: $\sigma_m = K_f\, \sigma_{m_o}$ & $\sigma_a = K_f\, \sigma_{a_o}$

➔ $\sigma_m = \sigma_a = 1.85(39.8) = 73.6\ MPa$

- The plot shows that the load line intersects the *Goodman* line:



$$S_a = \frac{rS_eS_{ut}}{rS_{ut} + S_e} = \frac{1(218.8)(620)}{1(620) + 218.8}$$

➔ $S_a = \boxed{162.4\ MPa} = S_m$

➔ $n_f = \dfrac{1}{\dfrac{\sigma_a}{S_e} + \dfrac{\sigma_m}{S_{ut}}}$ or $n_f = \dfrac{S_a}{\sigma_a} = \dfrac{S_m}{\sigma_m}$

➔ $n_f = \dfrac{162.4}{73.6} = \boxed{2.21}$