Failure Theories

1. Maximum Shear Stress (MSS) Theory for Ductile Materials

- Yielding **begins** whenever the maximum shear stress in any element **equals or exceeds** the maximum shear stress in a **tension test** specimen of the same material when that specimen begins to yield.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad or \qquad \sigma_1 - \sigma_3 = S_y$$

- Note that this implies that the yield strength in shear is given by S_{sy}=0.5S_y. (**HOW?**)
- For design purposes, a factor of safety (n) has to be used as shown below:

$$\tau_{max} = \frac{S_y}{2n} \qquad or \qquad \sigma_1 - \sigma_3 = \frac{S_y}{n}$$

- The principles stresses are labeled as σ_A and σ_B. Assume that $\boldsymbol{\sigma_A \geq \sigma_B}$, then the three cases to consider for the plane stress condition are:

Failure Theories

- Yielding **begins** whenever the maximum shear stress in any element **equals or exceeds** the maximum shear stress in a **tension test** specimen of the same material when that specimen begins to yield.

☺ **Case 1**: $\sigma_A \geq \sigma_B \geq 0$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$. The main equation reduces to;

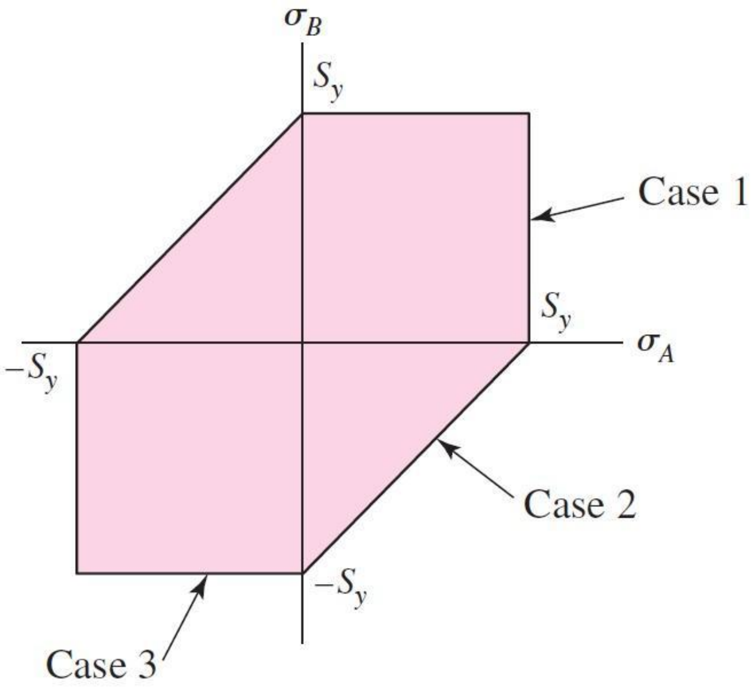
$$\sigma_A \geq S_y$$

☺ **Case 2**: $\sigma_A \geq 0 \geq \sigma_B$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$. The main equation becomes;

$$\sigma_A - \sigma_B \geq S_y$$

☺ **Case 3**: $0 \geq \sigma_A \geq \sigma_B$. For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$. The main equation gives;

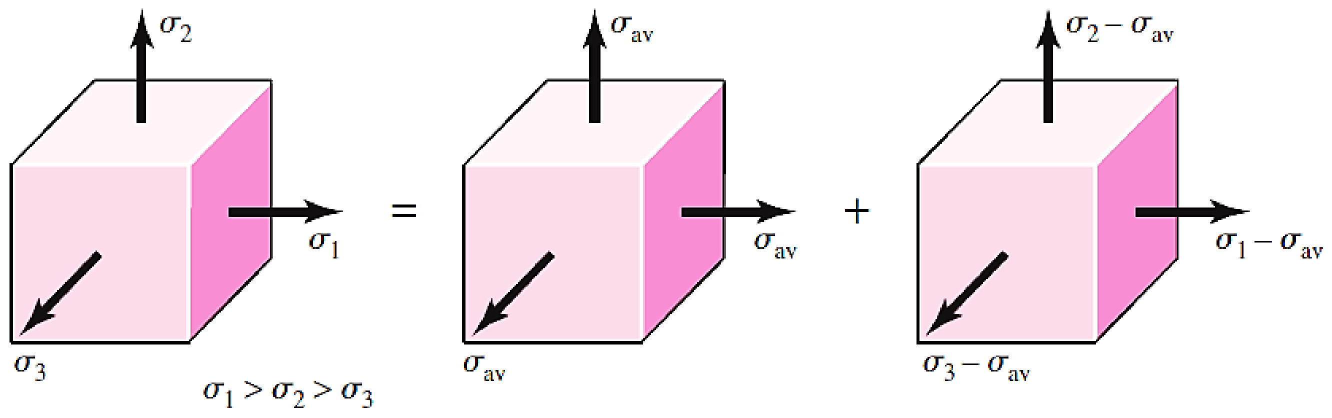
$$\sigma_B \leq -S_y$$



Failure Theories

2. Distortion Energy (DE) Theory for Ductile Materials

- Yielding occurs when the **distortion strain energy** per unit volume **reaches or exceeds** the distortion strain energy per unit volume for **yield** in simple **tension or compression** of the same material.



(*a*) Triaxial stresses (*b*) Hydrostatic component (*c*) Distortional component

pure volume change, no shape change

no volume change, just shape change

- The strain energy per unit volume for **simple tension** is:

$$u = \frac{1}{2}\epsilon\sigma$$

- For the element above the strain energy per unit volume is

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

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- The strain energy for producing only volume change (u_v)

$$u_v = \frac{3\sigma_{av}^2}{2E}(1-2v) \Rightarrow u_v = \frac{(1-2v)}{6E}[\sigma_1^2+\sigma_2^2+\sigma_3^2+2\sigma_1\sigma_2+2\sigma_2\sigma_3+2\sigma_3\sigma_1]$$

- The distortion energy is obtained by

$$u_d = \frac{1+v}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]$$

- For the simple tensile test, at yield, $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$,

$$u_d = \frac{1+v}{2E}S_y^2$$

- So for the general state of stress,

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \geq S_y$$

The left side of this equation is called effective stress $(\acute{\sigma})$, and usually is known von Misses stress.

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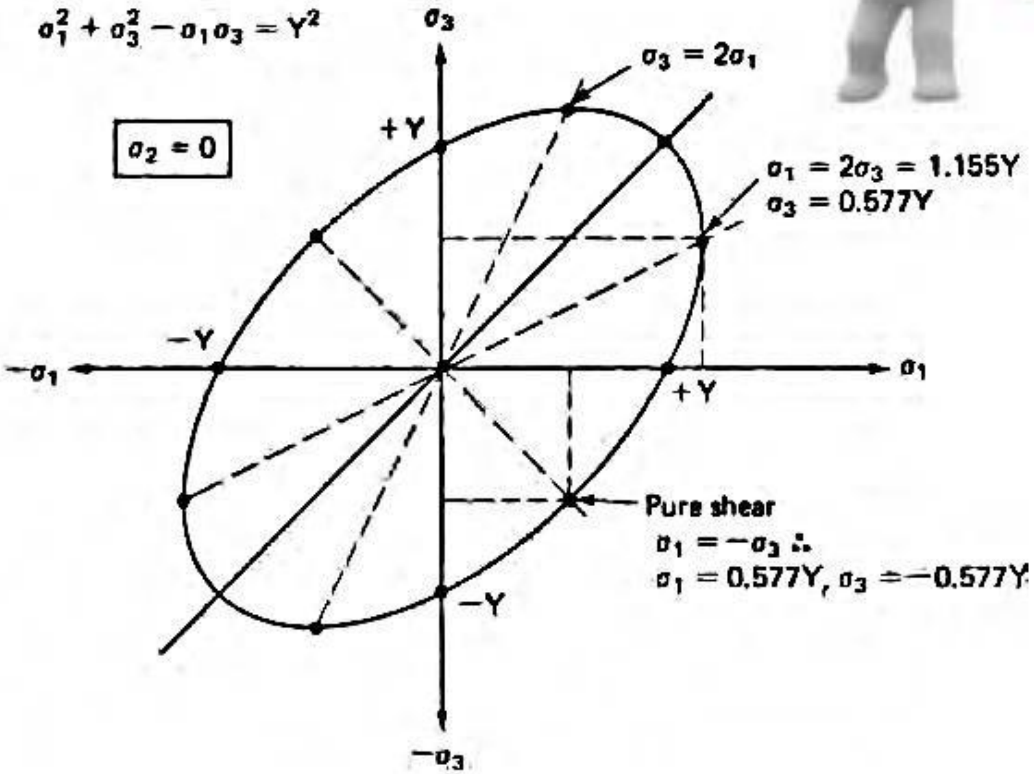
- Using xyz components of three-dimensional stress, the von Mises stress can be written as

$$\sigma' = \frac{1}{\sqrt{2}}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + \left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{1/2}$$

Consider a case of pure shear τ_{xy}, where for plane stress $\sigma_x = \sigma_y = 0$.

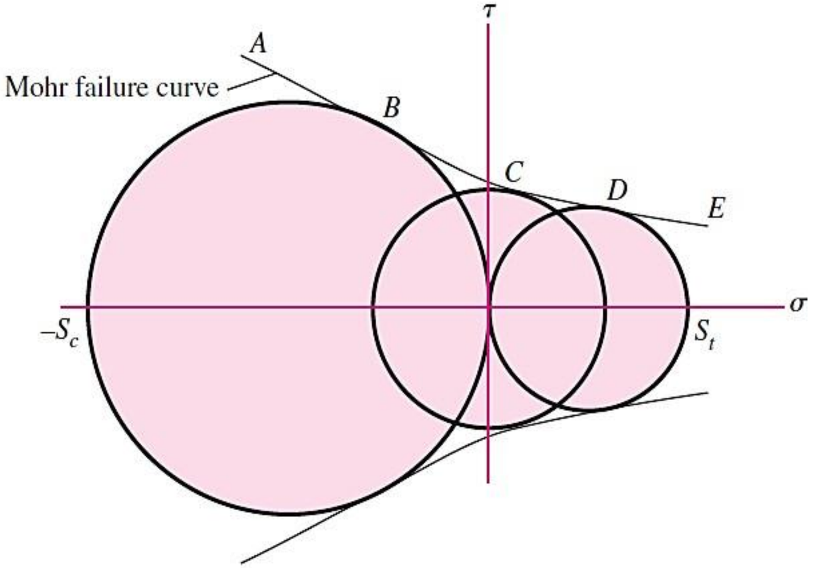
$$\left(3\tau_{xy}^2\right)^{1/2} = S_y \quad or \quad \tau_{xy} = \frac{S_y}{\sqrt{3}} = 0.577 S_y$$

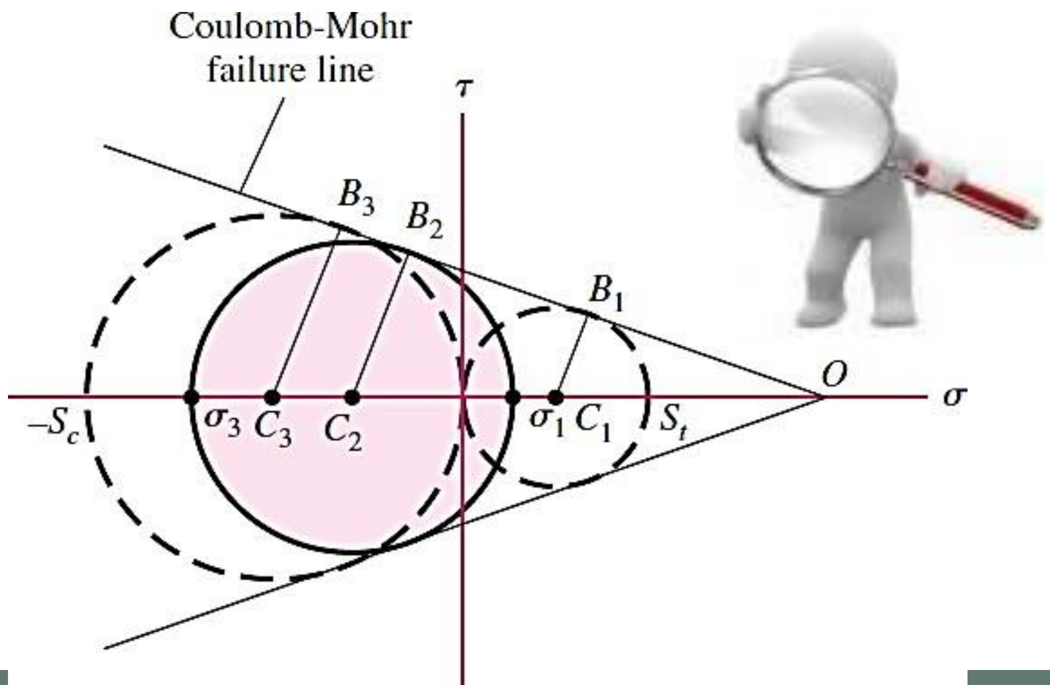


Failure Theories

3. Coulomb-Mohr Theory for Ductile Materials

- Not all materials have compressive strengths equal to their corresponding tensile values.
- Mohr's hypothesis was to use the results of tensile, compressive, and torsional shear tests to construct the three circles in the figure defining a failure envelope tangent to the three circles.





- A variation of Mohr's theory, called the Coulomb-Mohr theory or the internal-friction theory, assumes that the boundary BCD in the figure above is straight. With this assumption only the tensile and compressive strengths are necessary.

Failure Theories

- Consequently, the Coulomb-Mohr theory can be expressed as, $\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$ **How?**

- For plane stress, the failure conditions are

Case 1: $\sigma_A \geq \sigma_B \geq 0$. For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$. The above equation reduces to $\sigma_A \geq S_t$

Case 2: $\sigma_A \geq 0 \geq \sigma_B$. Here, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$. The above equation becomes $\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1$

Case 3: $0 \geq \sigma_A \geq \sigma_B$. Here, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$. The above equation reduces to $\sigma_B \geq -S_c$

- for design purposes, $\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n}$

- For pure shear τ, $\sigma_1 = -\sigma_3 = \tau$. The torsional yield strength occurs when $\boldsymbol{\tau_{max} = S_{sy}}$. Substituting $\sigma_1 = -\sigma_3 = S_{sy}$ into $S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}}$

Failure Theories

4. Maximum Normal Stress (MNS) Theory for Brittle Materials

- Failure occurs whenever one of the three principal stresses equals or exceeds the strength.

$$\sigma_1 \geq S_{ut} \quad \text{or} \quad \sigma_3 \leq -S_{uc}$$

where the principle stresses are arranged as $\sigma_1 > \sigma_2 > \sigma_3$

- For plane stress conditions,

$$\sigma_A \geq S_{ut} \quad \text{or} \quad \sigma_B \leq -S_{uc}$$

- The failure criteria equations can be converted to design equations.

$$\sigma_A = \frac{S_{ut}}{n} \quad or \quad \sigma_B = -\frac{S_{uc}}{n}$$

Failure Theories

5. Modifications of the Mohr Theory for Brittle Materials

- Two modifications of the Mohr theory for brittle materials: the Brittle-Coulomb-Mohr (BCM) theory and the modified Mohr (MM) theory.

➢ **Brittle-Coulomb-Mohr (BCM)**

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \geq \sigma_B \geq 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \geq 0 \geq \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \geq \sigma_A \geq \sigma_B$$

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5. Modifications of the Mohr Theory for Brittle Materials

➢ Modified Mohr (MM)

$$\sigma_A = \frac{S_{ut}}{n}$$

$$\sigma_A \geq \sigma_B \geq 0$$
$$\sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| \leq 1$$

$$\frac{(S_{uc} - S_{ut})\sigma_A}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n}$$

$$\sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

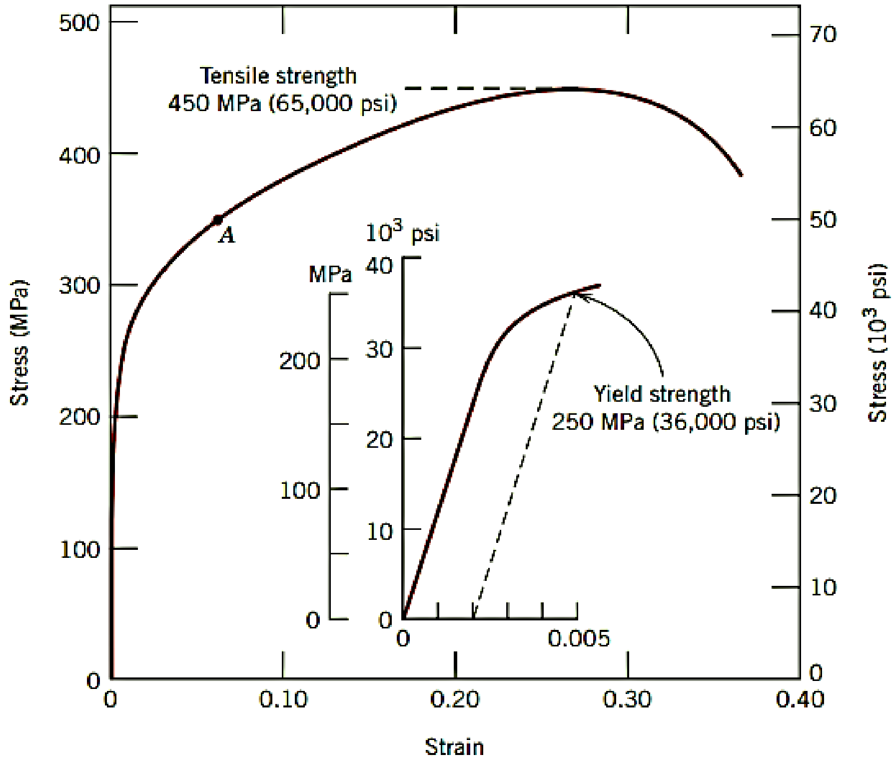
$$\sigma_B = -\frac{S_{uc}}{n}$$

$$0 \geq \sigma_A \geq \sigma_B$$

Example

From the tensile stress–strain behaviour for the brass specimen shown in Figure 2.4, determine the following:

(a) The modulus of elasticity
(b) The yield strength at a strain offset of 0.002
(c) The maximum load that can be sustained by a cylindrical specimen having an original diameter of 12.8 mm (0.505 in.)



Solution

(a) The modulus of elasticity is the slope of the elastic or initial linear portion of the stress–strain curve. The strain axis has been expanded in the inset to facilitate this computation. The slope of this linear region is the rise over the run, or the change in stress divided by the corresponding change in strain; in mathematical terms,

$$E = Slop = \frac{\Delta\sigma}{\Delta\varepsilon} = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1}$$

Inasmuch as the line segment passes through the origin, it is convenient to take both and as zero. If is arbitrarily taken as 150 MPa, then will have a value of 0.0016. Therefore,

$$E = \frac{150 - 0}{0.0016 - 0} = 93.8\ \text{MPa}$$

(b) The 0.002 strain offset line is constructed as shown in the inset; its intersection with the stress–strain curve is at approximately 250 MPa, which is the yield strength of the brass.

(c) The maximum load that can be sustained by the specimen is calculated as

$$F = \sigma . A_o = \sigma . \left(\frac{d_o}{2}\right)^2 \pi = (450 \times 10^6)\left(\frac{12.8 \times 10^{-3}}{2}\right)^2 \pi = 57900\ \text{N}$$

(d) To compute the change in length, it is first necessary to determine the strain that is produced by a stress of 345 MPa. This is accomplished by locating the stress point on the stress–strain curve, point A, and reading the corresponding strain from the strain axis, which is approximately 0.06. Inasmuch as lo=250 mm, we have

$$\Delta l = \varepsilon l_o = (0.06)(250) = 15\text{mm}$$

Example

A solid shaft of diameter d is made of AISI 1020 steel (as rolled) and is subjected to a tensile axial force of 200kN and a torque of 1.5kN.m.

(a) What is the safety factor against yielding if the diameter is 50mm?

(b) For the situation of (a), what adjusted value of diameter is required to obtainsafety factor against yielding of 2 ?

Solution:

(a) $$\sigma_x = \frac{P}{A} = \frac{4P}{\pi d^2}, \qquad \tau_{xy} = \frac{Tc}{J} = \frac{T(d/2)}{\pi\, d^2/32} = \frac{16T}{\pi d^3}$$

Note that σ_x is uniformly distributed, and τ_{xy} is evaluated at the surface of the shaft. Therefore, we have a state of plane stress with $\sigma_y = 0$. The principle normal stresses are $\sigma_3 = \sigma_y = 0$ and

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{2P}{\pi d^2} \pm \sqrt{\left(\frac{2P}{\pi d^2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

The effective stress is

$$\acute{\sigma} = \sqrt{(1/2)[(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2]} = 2\sqrt{\left(\frac{2P}{\pi d^2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2} = \frac{4}{\pi d^2}\sqrt{P^2 + \left(\frac{8T}{d}\right)^2} = 159.1\text{MPa}$$

Substituting P=200000N, T=1.5x106 N.mm and d=50mm and take the yield strength of the material σ_o =260MPa

$n = \sigma_o / \sigma = (260\text{MPa})/(159.1\text{MPa}) = 1.63$

(b) H.W.