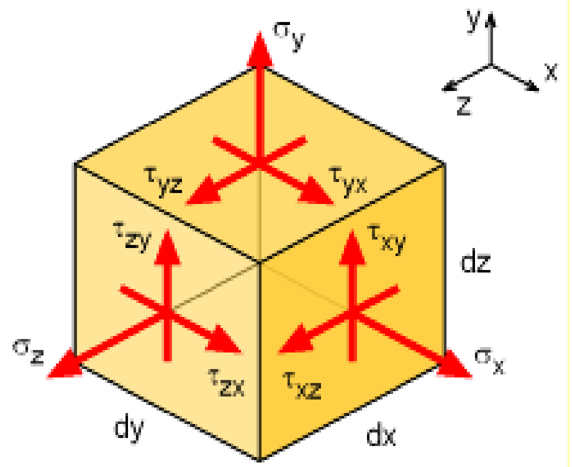
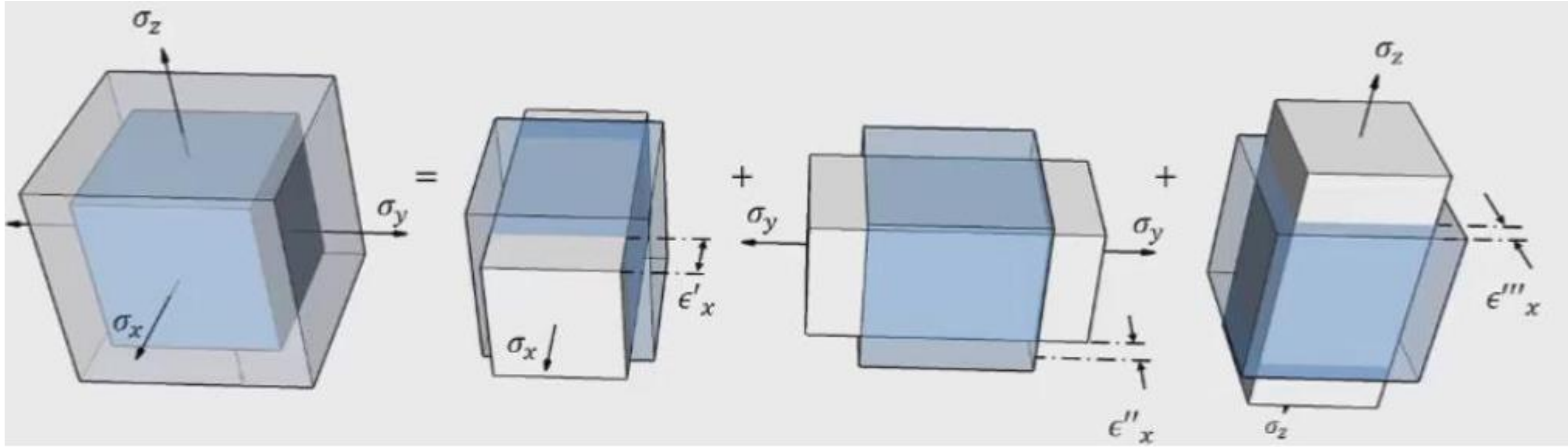
Complex Stresses

- The most general state of stresses is the **complex stresses** represented by the unit cube shown here.
- It is assumed that shear stresses can generate only shear strains. Thus, the longitudinal strains are produced exclusively by the normal stresses.





$$\varepsilon'_x = \frac{\sigma_x}{E} \qquad \varepsilon''_x = -\upsilon\frac{\sigma_y}{E} \qquad \varepsilon'''_x = -\upsilon\frac{\sigma_z}{E}$$

Hooke's law:

$$\varepsilon = \frac{\sigma_x}{E}$$

Poisson's ratio:

$$\upsilon = -\frac{\varepsilon_{lat}}{\varepsilon_{log}}$$

Complex Strains

$$\varepsilon_x = \varepsilon_x' + \varepsilon_x'' + \varepsilon_x''' = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)]$$

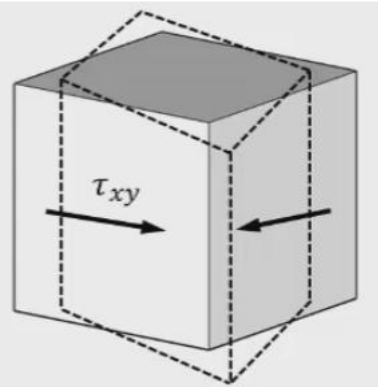
Using the same procedure results in

$$\varepsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_x + \sigma_z)]$$

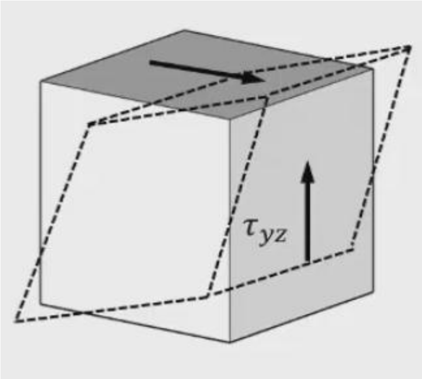
and

$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_x + \sigma_y)]$$

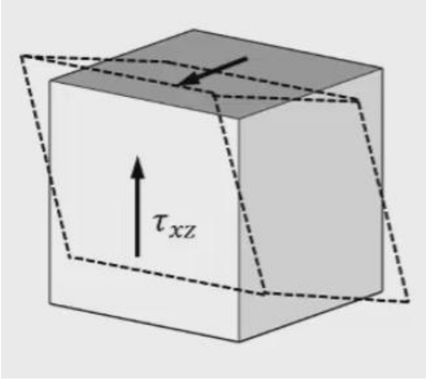
✓ Shear strains cause deformation only, and will not cause other strains in the material.



$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

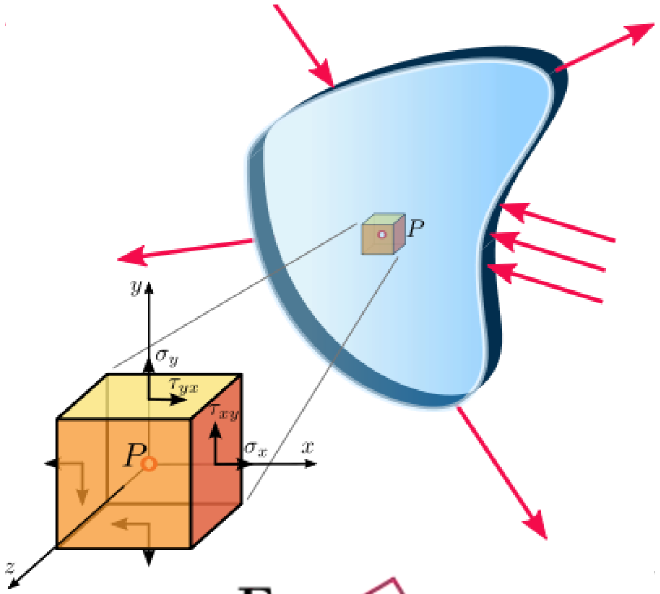


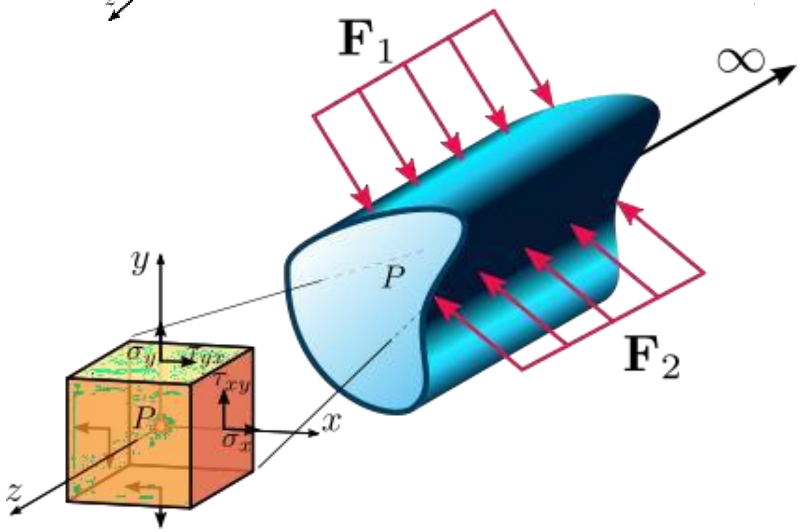
$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$



$$\gamma_{xz} = \frac{1}{G}\tau_{xz}$$

Complex Strains



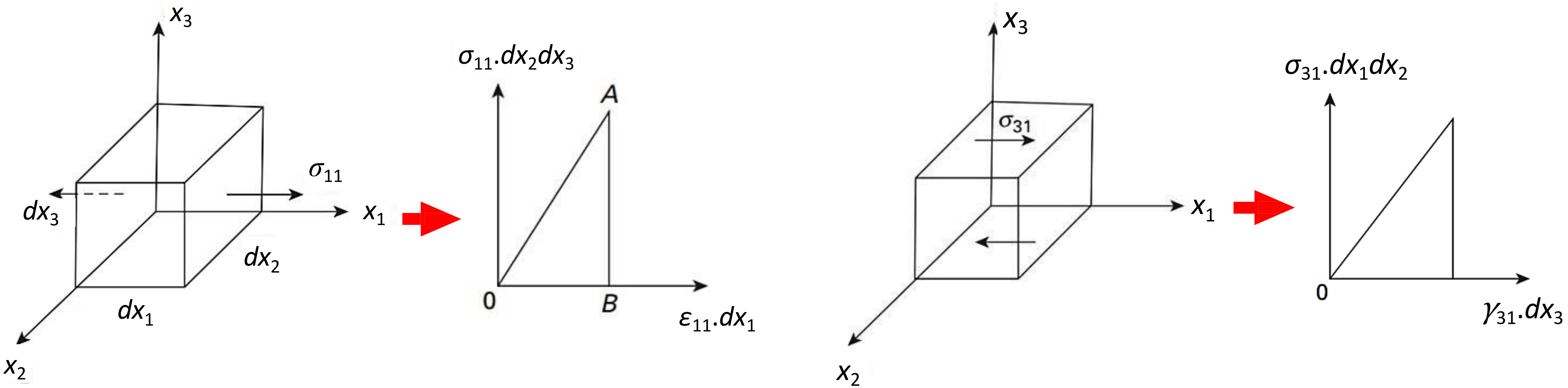


- In sheets and plates, the state of stress can be assumed to be bidimensional because normal stresses (normal to the surface) are **zero** at the surface. This state of stress is also known as **plane stress.**
- If **one dimension is infinite** with respect to the other two, strain in it is constrained zero; hence, one has two dimensions left. This state is called bidimensional or, more commonly, **plane strain.**
- Plane strain occurs when strain is constrained zero in one direction by some other means.

Strain energy (Deformation energy)

- The **work done** (W) is converted into **heat** (Q) and an increase in **internal energy** (U).
- For most solids, the elastic work produces an **insignificant** amount of heat. Hence, the **work done** on a body during deformation is **converted** into **internal energy** (strain energy).
- The **work done** is given by product of **force** and **change in length** (see the figure).

$$W = \frac{1}{2}[\sigma_{11}(dx_2 dx_3)\varepsilon_{11} dx_1] = \frac{1}{2}[\sigma_{11}\varepsilon_{11}(dx_1 dx_2 dx_3)]$$



Strain energy (Deformation energy)

The **work done** (W) per unit volume is: $$W_{11} = \frac{1}{2}\,\sigma_{11}\varepsilon_{11}$$

- for the **shear stress**, σ_{31}, the work done per unit volume is: $$W_{31} = \frac{1}{2}\,\sigma_{31}\varepsilon_{31}$$

- for **two or more stresses**, the total work done per unit volume or the strain energy density is:

$$W = \frac{1}{2}\,[\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\gamma_{12} + 2\sigma_{23}\gamma_{23} + 2\sigma_{31}\gamma_{31}]$$

- In more compact indicial notation, we can write; $$W = \frac{1}{2}\,\sigma_{ij}\varepsilon_{ij}$$

- For a linearly elastic solid under a **uniaxial stress** we can use the **Hooke's law** to obtain;

$$U = W = \frac{1}{2E}\,\sigma_{ij}^{2}$$

Anisotropic Effects

- Real materials are **never** perfectly isotropic
- In highly **anisotropic** materials, any one component of stress can cause strains in all six components.
- In the general 3-D, there are nine components of stress and a corresponding nine components of **strain**.

Stress components

$$\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{bmatrix}$$

Strain components

$$\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{bmatrix} = \begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}$$

- The off-diagonal components are equal, i.e., $\sigma_{13} = \sigma_{31}$, $\sigma_{12} = \sigma_{21}$ and $\sigma_{23} = \sigma_{32}$ as well as $\varepsilon_{13} = \varepsilon_{31}$, $\varepsilon_{12} = \varepsilon_{21}$ and $\varepsilon_{23} = \varepsilon_{32}$ as can be seen above.

Anisotropic Effects

- The indices can be replaced into;

11⇨ 1	23⇨ 4
22⇨ 2	13⇨ 5
33⇨ 3	12⇨ 6

therefore we have $\begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix}$ and $\begin{bmatrix} \varepsilon_1 & \varepsilon_6/2 & \varepsilon_5/2 \\ \varepsilon_6/2 & \varepsilon_2 & \varepsilon_4/2 \\ \varepsilon_5/2 & \varepsilon_4/2 & \varepsilon_3 \end{bmatrix}$

Because we have

$$\varepsilon_4 = 2\varepsilon_{23} = \gamma_{23}$$
$$\varepsilon_5 = 2\varepsilon_{13} = \gamma_{13}$$
$$\varepsilon_6 = 2\varepsilon_{12} = \gamma_{12}$$

why ?

- For the generalized case, Hooke's law may be expressed as:

$$\sigma_i = C_{ij}.\varepsilon_j$$
$$\varepsilon_i = S_{ij}.\sigma_j$$

Anisotropic Effects

- In matrix format, the stress-strain relation showing the **36** (6 x 6) independent components of stiffness

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}$$

- In matrix format, the stress-strain relation showing the 36 (6 x 6) independent components of stiffness, which means $C_{ij}=C_{ji}$ and $S_{ij}=S_{ji}$

- Therefore, the 36 components (6x6) for the general **anisotropic** linear elastic solid are reduced to **21**. In **Isotropic** case, the elastic constants are reduced from **21** to **2**.

Anisotropic Effects

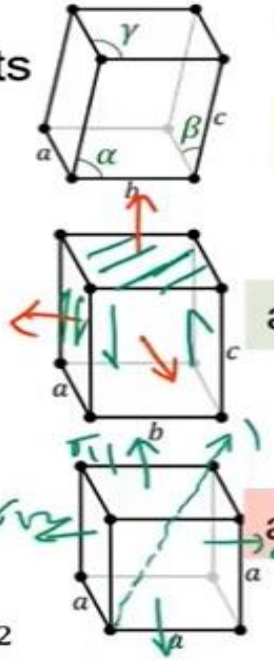
Shear decoupling → normal stress ✗ shear strains
shear strain in a plane can't be due to shear stress in another plane

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{22} & C_{23} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & & & & C_{55} & 0\\ & & & & & C_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{pmatrix}$$

4 $\frac{360}{4} = 90°$ retain symmetry

n-Fold means rotation of (360/n) retains the symmetry

Crystal Structure	Rotational Symmetry	Independent Elastic constants		
Triclinic	None	21		a ≠ b ≠ c and α ≠ β ≠ γ
Orthorhombic	2 perpendicular 2fold rotations axis	9		a ≠ b ≠ c and α = β = γ= 90°
Cubic	4 Three-fold	3		a = b = c and α = β = γ = 90°



σ_{11} C_{11} ε_{11} σ_{22} C_{22} ε_{22}

$C_{11} = C_{22} = C_{33}$ $C_{44} = C_{55} = C_{66}$ $C_{23} = C_{13} = C_{12}$

Anisotropic Effects

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ . & C_{11} & C_{12} & 0 & 0 & 0 \\ . & . & C_{11} & 0 & 0 & 0 \\ . & . & . & C_{44} & 0 & 0 \\ . & . & . & . & C_{44} & 0 \\ . & . & . & . & . & C_{44} \end{bmatrix}$$

- The number of independent elastic constants in a cubic system is three.
- For **isotropic** materials there are two independent constants because: $C_{44} = \frac{C_{11} - C_{12}}{2}$

  where: $C_{11} = \frac{E(1-v)}{(1+v)(1-2v)}$ $C_{12} = \frac{Ev}{(1+v)(1-2v)}$ $C_{44} = G = \frac{E}{2(1+v)}$

- For **anisotropic** systems, the equation of C_{44} does not apply, and we define an anisotropy ratio: $A = \frac{2C_{44}}{C_{11} - C_{12}} \neq 1$
- For the elastic compliances, we have, for the anisotropic case

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ . & S_{11} & S_{12} & 0 & 0 & 0 \\ . & . & S_{11} & 0 & 0 & 0 \\ . & . & . & S_{44} & 0 & 0 \\ . & . & . & . & S_{44} & 0 \\ . & . & . & . & . & S_{44} \end{bmatrix}$$

- For isotropic materials $S_{44} = 2(S_{11} - S_{12})$

Anisotropic Effects

- Young's modulus: $$E = \frac{1}{S_{11}}$$
- Rigidity or shear modulus: $$G = \frac{1}{2(S_{11} - S_{12})} = \frac{1}{S_4}$$
- Compressibility (B) and bulk modulus (K:($$B = \frac{1}{K} = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{-\frac{1}{3}(\sigma_{11} + \sigma_{12} + \sigma_{13})}$$
- Poisson's ratio: $$v = -\frac{S_{12}}{S_{11}}$$
- Lame's constants: $$\mu = C_{44} = \frac{1}{2}(C_{11} - C_{12}) = \frac{1}{S_{44}} = G$$ $$\lambda = C_{12} = \frac{Ev}{(1+v)(1-2v)}$$

Expressing the **strains as function** of stresses for isotropic materials

$$\varepsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{12}\sigma_3 = \frac{1}{E}[\sigma_1 - v(\sigma_2 + \sigma_3)]$$

$$\varepsilon_2 = S_{12}\sigma_1 + S_{11}\sigma_2 + S_{12}\sigma_3 = \frac{1}{E}[\sigma_2 - v(\sigma_1 + \sigma_3)]$$

$$\varepsilon_3 = S_{12}\sigma_1 + S_{12}\sigma_2 + S_{11}\sigma_3 = \frac{1}{E}[\sigma_3 - v(\sigma_1 + \sigma_2)]$$

$$\varepsilon_4 = \frac{1}{2}(S_{11} - S_{12})\sigma_4 = \frac{1}{G}\sigma_4$$

$$\varepsilon_5 = \frac{1}{2}(S_{11} - S_{12})\sigma_5 = \frac{1}{G}\sigma_5$$

$$\varepsilon_6 = \frac{1}{2}(S_{11} - S_{12})\sigma_6 = \frac{1}{G}\sigma_6$$

Anisotropic Effects

Expressing the **stresses as function** of strains for isotropic materials

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{12}\varepsilon_3 = (2\mu + \lambda)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$$

$$\sigma_2 = C_{12}\varepsilon_1 + C_{11}\varepsilon_2 + C_{12}\varepsilon_3 = \lambda\varepsilon_1 + (2\mu + \lambda)\,\varepsilon_2 + \lambda\varepsilon_3$$

$$\sigma_3 = C_{12}\varepsilon_1 + C_{12}\varepsilon_2 + C_{11}\varepsilon_3 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (2\mu + \lambda)\varepsilon_3$$

$$\sigma_4 = \frac{1}{2}(C_{11} - C_{12})\varepsilon_4 = \mu\varepsilon_4$$

$$\sigma_5 = \frac{1}{2}(C_{11} - C_{12})\varepsilon_5 = \mu\varepsilon_5$$

$$\sigma_6 = \frac{1}{2}(C_{11} - C_{12})\varepsilon_6 = \mu\varepsilon_6$$

Elastic Properties of Ceramics

- **Porosity** is the crucial parameters on which the elastic modulus of ceramics strongly depends
- Wachtman and MacKenzie expression for the elastic modulus of ceramics as a **function of porosity** is:

$$E = E_o \left(1 - f_1 p - f_2 p^2\right)$$

.are constants $_2 f$ and $_1 f$ is the porosity and p where

- For **spherical voids** and For relatively **low porosity**,

$$E = E_o(1 - 1.9p)$$

- The presence of **microcracks**, which affect the elastic properties of ceramics, **decrease** the stored elastic energy and **reduce** the effective Young's modulus.

Elastic Properties of Ceramics

- The expression developed by Salganik explains how the presence of microcracks change the Young modulus of ceramics;

$$\frac{E}{E_o} = [1 + \frac{16(10 - 3v_o^2)(1 - v_o^2)}{45(2 - v_o)} Na^3]^{-1} = (1 + ANa^3)^{-1}$$

where a is the radius of a mean crack, and N is the number of cracks per unit volume.

- O'Connell and Budiansky arrived at a slightly different expression

$$\frac{E}{E_o} = 1 - \frac{16(10 - 3v)(1 - v^2)}{45(2 - v)} f_s$$

- where f_s is the volume fraction of cracks and v is Poisson's ratio of the porous material

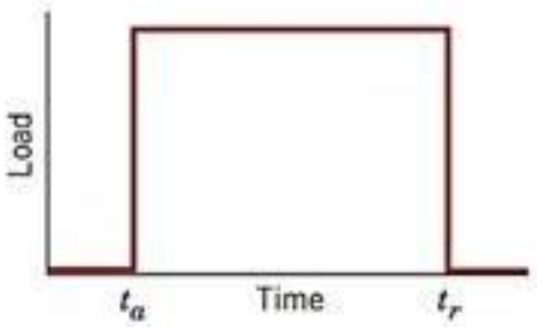
$$v = v_o \left(1 + \frac{16 f_s}{9}\right)$$

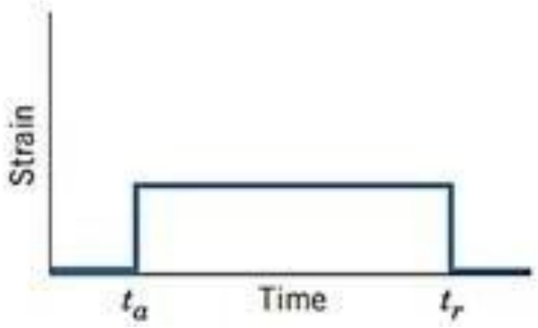
Elastic Properties of Polymers

- Elastic behaviour of polymeric materials is strongly dependent on both **temperature** and **time**.
- The behaviour is called viscoelastic or anelastic.
- In most polymers, there are dramatic changes in ***E*** between 20° C and 200°C, for most metals and ceramics, the changes in ***E*** in this range can be neglected.
- **Above** glass transition temperature T_g, ***E*** is considerably **low**, and the behaviour of the polymer can be described as rubbery and viscous. **Below** T_g, ***E*** is considerably **higher**, and the behaviour is closer to **linear elastic**.

Anelasticity (Viscoelastic behaviour)

- Elastic deformation is time independent, and upon release of the load the strain is totally recovered (see the figure.(

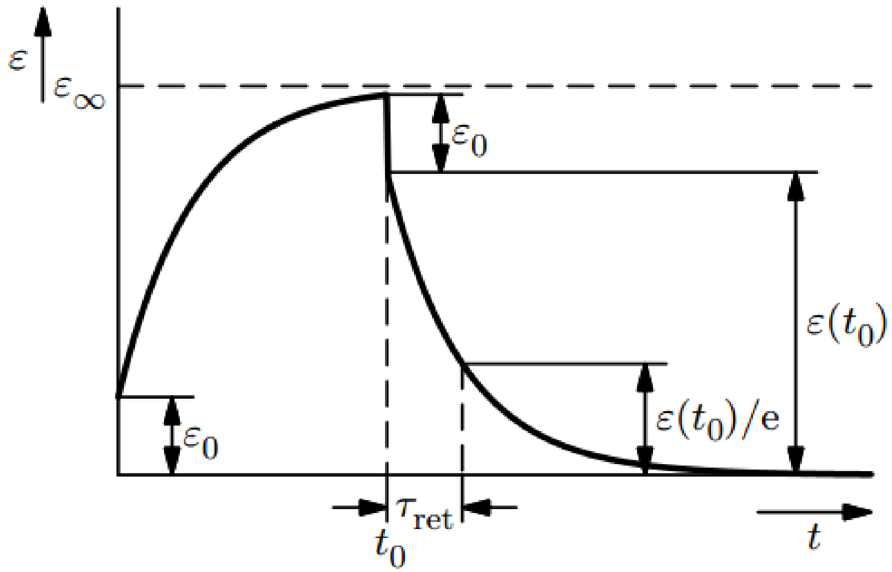




- If the stress in a polymer is raised abruptly from zero to a value σ ,the strain increases instantaneously to a value ε_0 (as time independent case), but then it further increases with time.

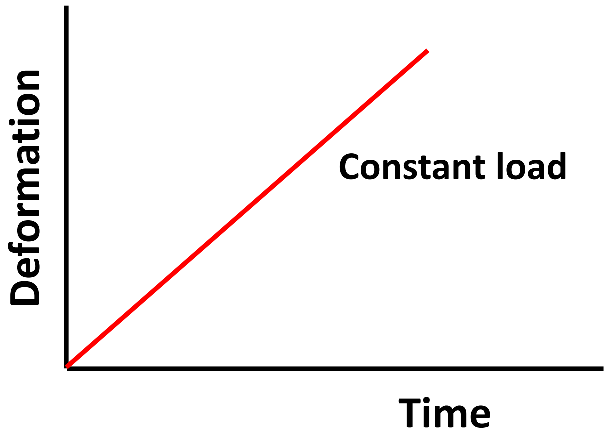
- the time-dependent Young's modulus at constant stress $E_c\,(t)$:

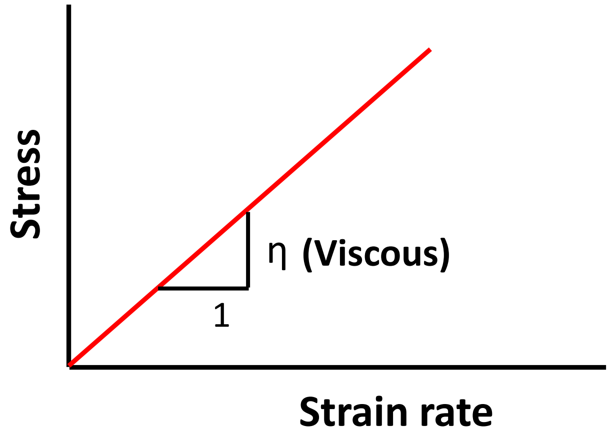
$$E_c(t) = \frac{\sigma}{\varepsilon(t)}$$

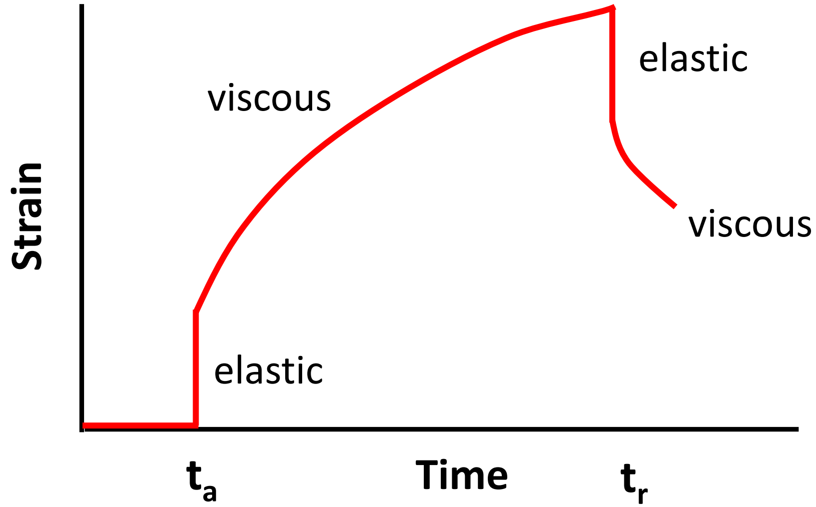


Anelasticity (Viscoelastic behaviour)

- Behaviour of most polymers is in **between** behavior of elastic and viscous materials.
  1. At low temperature and high strain rate, polymer demonstrate **elastic behavior**.
  2. At high temperature and low strain rate, polymer demonstrate **viscous behavior**.
  3. At intermediate temperature and strain rate, polymer demonstrate **viscos-elastic behavior**.
- Instantaneously elastic strain followed by viscous time depended strain.







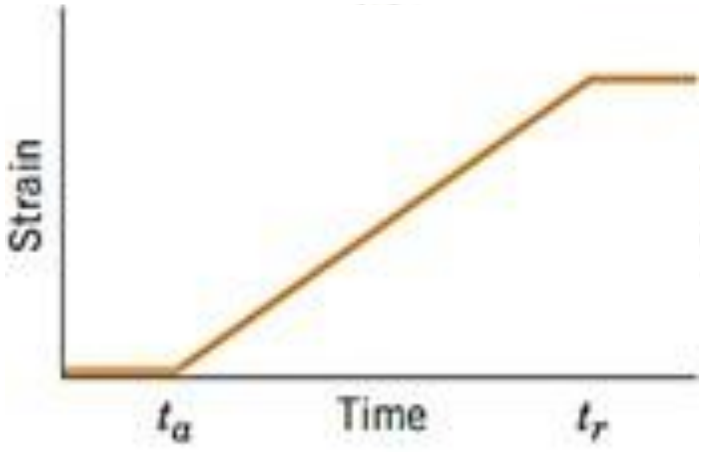
$$\sigma = \eta \frac{de}{dt}$$

Anelasticity (Viscoelastic behaviour)

- If the load is removed after a time t_o, the strain decreases instantaneously by the time-independent strain ε_o and then reduces slowly to zero, which is called **intermediate viscoelastic behavior**.

- The **retardation time** τ_{ret} is defined as the time needed to reduce the time-dependent part of the strain by a factor 1/e.

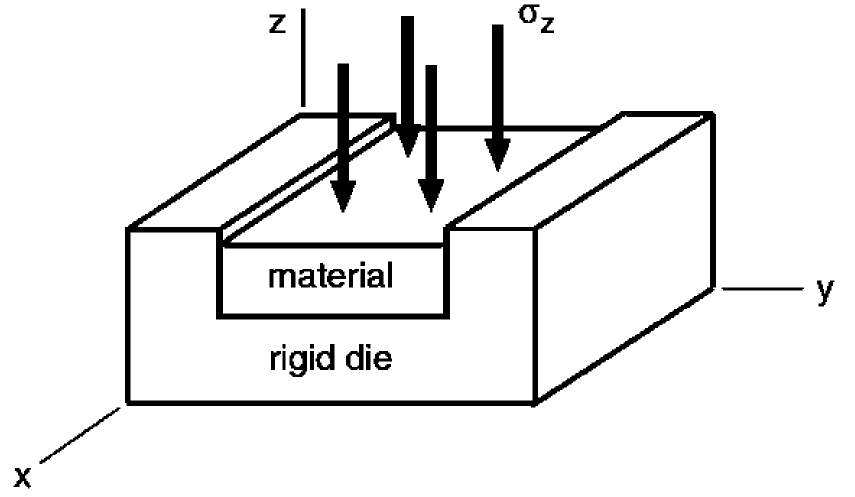
- For **totally viscous behaviour**, deformation or strain is not instantaneous; in response to an applied stress, deformation is delayed or dependent on time (see the figure here).



- This deformation is **not reversible** or completely recovered after the stress is released.

Example:

A sample of material subjected to a compressive stress σ_z is confined so that it cannot deform in the y-direction, as shown in the figure. Assume that there is no friction against the die, so that deformation can freely occur in the x-direction. Assume further that the material is isotropic and exhibits linear-elastic behavior. Determine the following in terms of σ_z and the elastic constants of the material:



a. The stress that develops in the *y*-direction.
b. The strain in the *z*-direction.
c. The strain in the *x*-direction.
d. The stiffness $E' = \sigma_z / \varepsilon_z$ in the *z*-direction. Is this apparent modulus equal to the elastic modulus *E* from a uniaxial test on the material? Why or why not?

e. Assume that the compressive stress in the *z*-direction has a magnitude of 75MPa and that the block is made of a copper alloy, and then calculate σ_y , ε_z , ε_x and E'.

Solution:

The situation posed requires substituting $\varepsilon_y = 0$ and $\sigma_x = 0$, and also treating σ_z as a known quantity.

a. The stress in the y-direction is obtained from:

$$0 = \frac{1}{E}[\sigma_y - v(0 + \sigma_z)], \qquad \sigma_y = v\sigma_z$$

b. The strain in the z-direction is given by using this σ_y

$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(0 + v\sigma_z)], \qquad \varepsilon_z = \frac{1 - v^2}{E}\sigma_z$$

c. The strain in the x-direction is given by using the value of σ_y obtained in (a)

$$\varepsilon_x = \frac{1}{E}[0 - v(v\sigma_z + \sigma_z)], \qquad \varepsilon_x = -\frac{v(1 + v)}{E}\sigma_z$$

d. The apparent stiffness in the z-direction is obtained immediately from the equation for ε_z

$$E' = \frac{\sigma_z}{\varepsilon_z} = \frac{E}{1 - v^2}$$

Solution:

For the copper alloy, Table 5.2 provides constants, *E*=130 GPa, and *V*=0.343. The compressive stress requires that a negative sign be applied, so that σ_z =−75 MPa. Substituting these quantities into the equations previously derived gives

$$\sigma_y = v\sigma_z = (0.343)(-75 \text{ MPa}) = -25.7 \text{ MPa}$$

$$\varepsilon_z = \frac{1-v^2}{E}\sigma_z = \frac{1-0.343^2}{130000}(-75) = -509\times10^{-6}$$

$$\varepsilon_z = -\frac{v(1+v)}{E}\sigma_z = -\frac{0.343(1+0.343)}{130000}(-75) = 266\times10^{-6}$$

$$E' = \frac{E}{1-v^2} = \frac{130000}{1-0.343^2} = 147300 \text{ MPa}$$