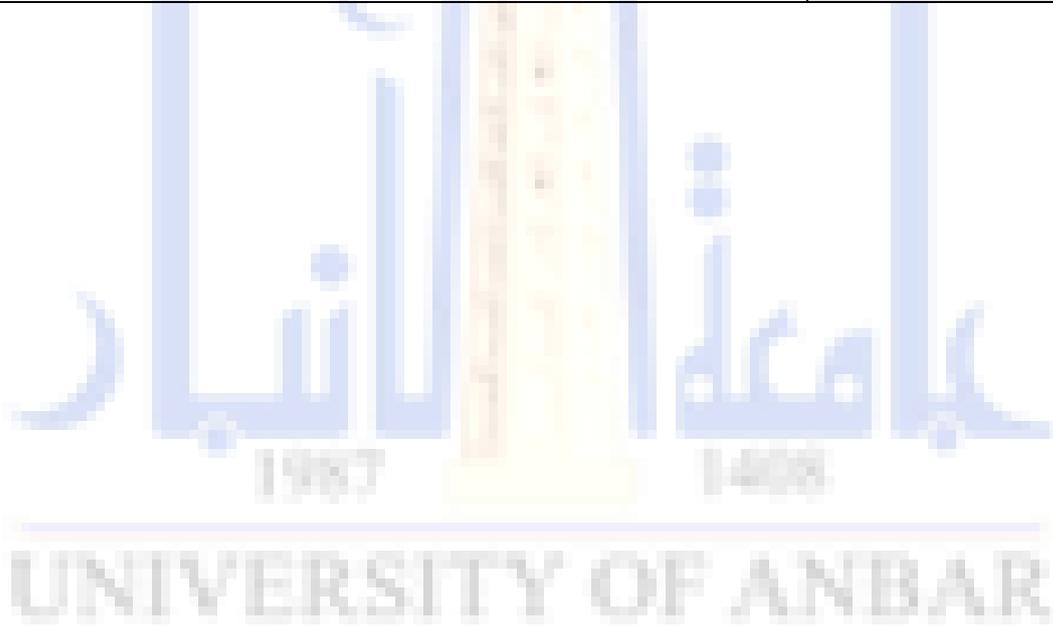


العلوم	الكلية
الرياضيات	القسم
Complex Analysis 1	المادة باللغة الانجليزية
التحليل العقدي 1	المادة باللغة العربية
المرحلة الرابعة	المرحلة الدراسية
عمر كريم علي حسون	اسم التدريسي
Complex plan	عنوان المحاضرة باللغة الانجليزية
الفضاء المركب	عنوان المحاضرة باللغة العربية
7	رقم المحاضرة
Complex variables with an introduction Murray R. Spiegel.	المصادر والمراجع
Applied complex Analy with partial Differential Equations , Nakhle H Asmar (2002).	



## 2- Complex Plane

**Definition:** The complex plane  $\mathbb{C}$  is the set of all ordered pairs  $(a, b)$  of real numbers, with addition and multiplication defined by:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$\text{and } (a, b)(c, d) = (ac - bd, bc + ad)$$

**Remark:** A complex number  $z = x + iy$  is uniquely determined by an ordered pair of real numbers  $(x, y)$ .

For example,  $z = 3 - 9i = (3, -9)$ ,

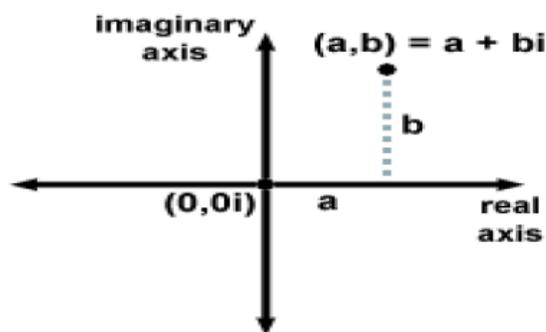
$$z = 8 = (8, 0),$$

$$z = i = (0, 1),$$

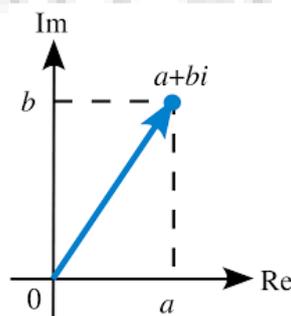
$$z = -7i = (0, -7).$$

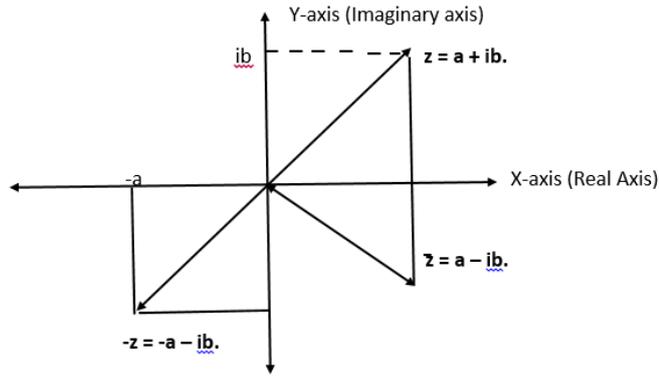
**Remark:** The coordinate plane is called the **complex plane** or simply the  $z$ -plane.

The horizontal or  $x$ -axis is called the **real axis** because each point on that axis represents a real number. The vertical or  $y$ -axis is called the **imaginary axis** because a point on that axis represents a pure imaginary number.



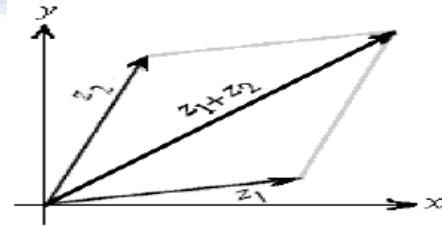
**Definition:** The modulus of a complex number  $z = a + ib$ , is the real number  $|z| = \sqrt{a^2 + b^2}$ . The modulus  $|z|$  of a complex number  $z$  is also called the **absolute value** of  $z$ .





**Properties:**

1.  $z \cdot \bar{z} = |z|^2$
2.  $|\bar{z}| = |z|$
3.  $|z|^2 = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$
4.  $|z| \geq |\text{Re}(z)| \geq \text{Re}(z)$  and  $|z| \geq |\text{Im}(z)| \geq \text{Im}(z)$
5.  $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
6.  $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
7.  $|z_1 + z_2| \leq |z_1| + |z_2|$  triangle inequality
8.  $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$  تبرهن بالاستقراء الرياضي  
i.e.,  $|\sum_{k=1}^n z_k| \leq \sum_{k=1}^n |z_k|$
9.  $|z_1 - z_2| \geq |z_1| - |z_2|$
10.  $||z_1| - |z_2|| \leq |z_1 - z_2|$



**Proof (7):**

$$\begin{aligned}
 |z_1 + z_2|^2 &= (z_1 + z_2) \cdot \overline{(z_1 + z_2)} = (z_1 + z_2) \cdot (\bar{z}_1 + \bar{z}_2) \\
 &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\
 &= |z_1|^2 + z_1\bar{z}_2 + \overline{z_2z_1} + |z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) \\
 &\leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| = |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |\bar{z}_2| \\
 &= |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| = (|z_1| + |z_2|)^2
 \end{aligned}$$

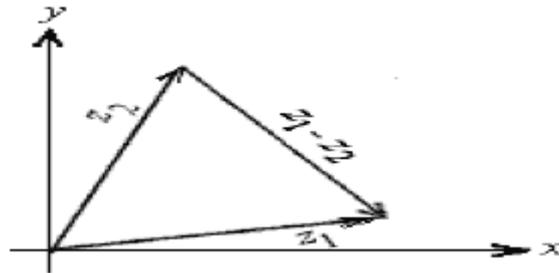
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Therefore  $|z_1 + z_2| \leq |z_1| + |z_2|$

**Remark:**  $|z_1 - z_2|$  is the distance in the plane on the complex number  $z_1$  from the complex number  $z_2$ .

If  $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



Proof (9):

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

$$\rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$$

Proof (10):

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \dots \dots \dots 1$$

$$|z_2| = |z_2 + z_1 - z_1| \leq |z_1 + z_2| + |-z_1| = |z_1 + z_2| + |z_1|$$

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

$$|z_1| - |z_2| \geq -|z_1 + z_2| \dots \dots \dots 2$$

From (1) and (2) we get  $-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$

Therefore  $||z_1| - |z_2|| \leq |z_1 + z_2|$

**Remark:**  $z_1 < z_2$  in general has no meaning but  $|z_1| < |z_2|$  means the point corresponding to  $z_1$  is closer to the origin than the point corresponding to  $z_2$ .

**Notation:** Let  $z_0 \in \mathbb{C}, r > 0$

$S_r(z_0) = \{z \in \mathbb{C}: |z - z_0| = r\}$  is the **circle** دائرة with center  $z_0$  and radius  $r$ .

$B_r(z_0) = \{z \in \mathbb{C}: |z - z_0| < r\}$  is the **ball** كرة with center  $z_0$  and radius  $r$ .

$D_r(z_0) = \{z \in \mathbb{C}: |z_1 - z_0| \leq r\}$  is the **disk** قرص with center  $z_0$  and radius  $r$ .

**Example:** The equation  $|z - 1 + 3i| = 2$  represents the circle whose center is  $z_0 = (1, -3)$  and whose radius is  $r = 2$ .

**Exercises:**

1. Find the modulus (absolute value) and conjugate of each of the following:

- $(1 - i)^2$

- $i(2 - i) - 4\left(1 + \frac{1}{4}i\right)$
- $\frac{2i}{3-4i}$
- $\frac{1-2i}{1+i} + \frac{2-i}{1-i}$
- $\frac{(4+3i)(1+i)}{7-i}$

2. Show that:

- $\overline{\bar{z} + 3i} = z - 3i$
- $\overline{\bar{z}} = -i\bar{z}$
- $\overline{(2 + i)^2} = 3 - 4i$
- $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$

3. Show that

- $\overline{\overline{z_1 z_2 z_3}} = \overline{z_1 z_2 z_3}$
- $\overline{z^4} = \bar{z}^4$

4. Find the complex number  $E$  which satisfies the following:

$$|E| = 1 \text{ and } \operatorname{Re}(E^2) = 0.$$

5. Show that equation  $|z - z_0| = R$  of circle, centered at  $z_0$  with radius  $R$ , can be written as:  $|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2$ .

6.  $z$  is real number if and only if  $z = \bar{z}$ .

7.  $z$  is either real or pure imaginary if and only if  $(\bar{z})^2 = z^2$ .

