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عنوان المحاضرة باللغة الانجليزية	**power series**
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المصادر والمراجع	Belmont, USA. ‹2008 ‹Calculus: Early Transcendentals, James Stewart
	Boston, USA. ‹2013 ‹Multivariable Calculus, Ron Larson and Bruce Edwards
	Thomas' Calculus, 11th Edition

Power Series

The series $\sum_{n=0}^{\infty} a_n(x-b)^n$ is called a power series in $(x-b)$, where a_n is a sequence in $\mathbb{R}$. When $b=0$, we say that $\sum_{n=0}^{\infty} a_n x^n$ is a power series.

Example: For what values of x does the series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$ converge? Apply the ratio test

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| < 1 \Rightarrow \left|\frac{(x-3)^{n+1}n!}{(n+1)!\,(x-3)^n}\right|$$
$$\Rightarrow \lim_{n\to\infty}\left|\frac{(x-3)^n(x-3)n!}{(n+1)n!\,(x-3)^n}\right|$$
$$\Rightarrow \lim_{n\to\infty}\left|\frac{(x-3)}{(n+1)}\right|$$
$$\Rightarrow |(x-3)|\lim_{n\to\infty}\frac{1}{n+1}$$
$$\Rightarrow |(x-3)|\times 0 = 0 < 1, \text{ for all } x$$

Thus, the series converge for all $x \in \mathbb{R}$.
Example: Find the series' interval of convergence for $\sum_{n=0}^{\infty}(\ln\ x)^n$? Apply the ratio test $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| < 1$

Example: Find the series' interval of convergence for $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^2}$?

Apply the ratio test $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| < 1$

$$\Rightarrow \left|\frac{(x+2)^{n+1}n2^n}{(n+1)2^{n+1}(x+2)^n}\right| < 1$$
$$\Rightarrow \frac{|(x+2)|}{2}\lim_{n\to\infty}\frac{n}{n+1} < 1$$
$$\Rightarrow \frac{|(x+2)|}{2} < 1$$
$$\Rightarrow -2 < x+2 < 2$$
$$\Rightarrow -4 < x < 0$$

Now, when $x=-4$ we have $\sum_{n=1}^{\infty}\frac{-1}{n}$ divergent series; when $x=0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, the alternating series which converges conditionally. Thus, the interval of convergence is $-4 < x \le 0$.

3.1 Taylor series and Maclaurin series

Suppose f is a given function which is k times differentiable at a given point $x = a$. Then, Taylor series is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^k(a)(x-a)^k}{k!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

When $a = 0$, in this case, the Taylor series is called Maclaurin series, and is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^k(0)x^k}{k!} + \cdots = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

Example: Find the Taylor series expansion of $f(x) = \sin(x)$ about the point $a = \frac{\Pi}{2}$?

We have

$$f(x) = \sin(x) \Rightarrow f\left(\frac{\Pi}{2}\right) = \sin\left(\frac{\Pi}{2}\right) = 1$$
$$f'(x) = \cos(x) \Rightarrow f'\left(\frac{\Pi}{2}\right) = \cos\left(\frac{\Pi}{2}\right) = 0$$
$$f''(x) = -\sin(x) \Rightarrow f''\left(\frac{\Pi}{2}\right) = -\sin\left(\frac{\Pi}{2}\right) = -1$$
$$f'''(x) = -\cos(x) \Rightarrow f'''\left(\frac{\Pi}{2}\right) = -\cos\left(\frac{\Pi}{2}\right) = 0$$
$$f^4(x) = \sin(x) \Rightarrow f^4\left(\frac{\Pi}{2}\right) = \sin\left(\frac{\Pi}{2}\right) = 1$$

and so on. Thus, the Taylor series is

$$f(x) = f\left(\frac{\Pi}{2}\right) + f'\left(\frac{\Pi}{2}\right)\left(x - \frac{\Pi}{2}\right) + \frac{f''\left(\frac{\Pi}{2}\right)\left(x-\frac{\Pi}{2}\right)^2}{2!} + \cdots + \frac{f^k\left(\frac{\Pi}{2}\right)\left(x-\frac{\Pi}{2}\right)^k}{k!} + \cdots$$
$$= 1 - \frac{(x-\Pi/2)^2}{2!} + \frac{(x-\Pi/2)^4}{4!} + \cdots$$

Example: Find the Maclaurin series expansion of $f(x) = \sin(x)$?
We have

$$\begin{aligned} f(x) &= \sin(x) && \Longrightarrow f(0) = \sin(0) = 0 \\ f'(x) &= \cos(x) && \Longrightarrow f'(0) = \cos(0) = 1 \\ f''(x) &= -\sin(x) && \Longrightarrow f''(0) = -\sin(0) = 0 \\ f'''(x) &= -\cos(x) && \Longrightarrow f'''(0) = -\cos(0) = -1 \\ f^4(x) &= \sin(x) && \Longrightarrow f^4(0) = \sin(0) = 0 \end{aligned}$$

and so on. Thus, the Maclaurin series is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^k(0)x^k}{k!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Example: Find the Maclaurin series expansion of $f(x) = \cos(x)$?

We have

$$\begin{aligned} f(x) = \cos(x) &\Longrightarrow f(0) = \cos(0) = 1 \\ f'(x) = -\sin(x) &\Longrightarrow f'(0) = -\sin(0) = 0 \\ f''(x) = -\cos(x) &\Longrightarrow f''(0) = -\cos(0) = -1 \\ f'''(x) = \sin(x) &\Longrightarrow f'''(0) = \sin(0) = 0 \\ f^4(x) = \cos(x) &\Longrightarrow f^4(0) = \cos(0) = 1 \end{aligned}$$

and so on. Thus, the Maclaurin series is

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^k(0)x^k}{k!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

Note that if $f(x) = \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, then

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n+1-1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)(2n)!}$$
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = cos(x)$$

Example: show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where $i = \sqrt{-1}$ is complex number by using power series?

Note that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
Also, we have $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1$, and so on.

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$
$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$
$$= \cos(\theta) + i\sin(\theta).$$