4.10 Inner Product Spaces

- In this section, we extend those concepts of $\mathbf{R}^n$ such as: dot product of two vectors, norm of a vector, angle between vectors, and distance between points, to general vector space.

- This will enable us to talk about the magnitudes of functions and orthogonal functions. This concepts are used to approximate functions by polynomials – a technique that is used to implement functions on calculators and computers.

- We will no longer be restricted to Euclidean Geometry, we will be able to create our own geometries on $\mathbf{R}^n$.

The dot product was a key concept on $\mathbf{R}^n$ that led to definitions of norm, angle, and distance. Our approach will be to generalize the dot product of $\mathbf{R}^n$ to a general vector space with a mathematical structure called an **inner product**. This in turn will be used to define norm, angle, and distance for a general vector space.

Definition

An **inner product** on a real space V is a function that associates a number, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, with each pair of vectors **u** and **v** of V. This function has to satisfy the following conditions for vectors **u**, **v**, and **w**, and scalar c.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (symmetry axiom)
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (additive axiom)
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$ (homogeneity axiom)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$
   (positive definite axiom)

A vector space V on which an inner product is defined is called an **inner product space**.

Any function on a vector space that satisfies the axioms of an inner product defines an inner product on the space.

There can be many inner products on a given vector space.

Example 1

Let $\mathbf{u} = (x_1, x_2)$, $\mathbf{v} = (y_1, y_2)$, and $\mathbf{w} = (z_1, z_2)$ be arbitrary vectors in $\mathbf{R}^2$. Prove that $\langle \mathbf{u}, \mathbf{v} \rangle$, defined as follows, is an inner product on $\mathbf{R}^2$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1 y_1 + 4 x_2 y_2$$

Determine the inner product of the vectors (–2, 5), (3, 1) under this inner product.

Solution

Axiom 1: $\langle \mathbf{u}, \mathbf{v} \rangle = x_1 y_1 + 4 x_2 y_2 = y_1 x_1 + 4 y_2 x_2 = \langle \mathbf{v}, \mathbf{u} \rangle$

Axiom 2:
$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= \langle (x_1, x_2) + (y_1, y_2),\ (z_1, z_2) \rangle \\
&= \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle \\
&= (x_1 + y_1) z_1 + 4(x_2 + y_2) z_2 \\
&= x_1 z_1 + 4 x_2 z_2 + y_1 z_1 + 4 y_2 z_2 \\
&= \langle (x_1, x_2), (z_1, z_2) \rangle + \langle (y_1, y_2), (z_1, z_2) \rangle \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Axiom 3: $\langle c\mathbf{u}, \mathbf{v} \rangle = \langle c(x_1, x_2), (y_1, y_2) \rangle$

$= \langle (cx_1, cx_2), (y_1, y_2) \rangle$

$= cx_1y_1 + 4cx_2y_2 = c(x_1y_1 + 4x_2y_2)$

$= c \langle \mathbf{u}, \mathbf{v} \rangle$

Axiom 4: $\langle \mathbf{u}, \mathbf{u} \rangle = \langle (x_1, x_2), (x_1, x_2) \rangle = x_1^2 + 4x_2^2 \geq 0$

Further, $x_1^2 + 4x_2^2 = 0$ if and only if $x_1 = 0$ and $x_2 = 0$. That is $\mathbf{u} = 0$.

Thus $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$.

The four inner product axioms are satisfied,

$\langle \mathbf{u}, \mathbf{v} \rangle = x_1y_1 + 4x_2y_2$ is an inner product on $\mathbf{R}^2$.

The inner product of the vectors (–2, 5), (3, 1) is

$$\langle (-2, 5), (3, 1) \rangle = (-2 \times 3) + 4(5 \times 1) = 14$$

Example 2

Consider the vector space M_{22} of 2×2 matrices. Let **u** and **v** defined as follows be arbitrary 2×2 matrices.

$$\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Prove that the following function is an inner product on M_{22}.

$$\langle \mathbf{u}, \mathbf{v} \rangle = ae + bf + cg + dh$$

Determine the inner product of the matrices $\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 2 \\ 9 & 0 \end{bmatrix}$.

Solution

Axiom 1: $\langle \mathbf{u}, \mathbf{v} \rangle = ae + bf + cg + dh = ea + fb + gc + hd = \langle \mathbf{v}, \mathbf{u} \rangle$

Axiom 3: Let k be a scalar. Then

$$\langle k\mathbf{u}, \mathbf{v} \rangle = kae + kbf + kcg + kdh = k(ae + bf + cg + dh) = k \langle \mathbf{u}, \mathbf{v} \rangle$$

$$\left\langle \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 9 & 0 \end{bmatrix} \right\rangle = (2 \times 5) + (-3 \times 2) + (0 \times 9) + (1 \times 0) = 4$$

Example 3

Consider the vector space P_n of polynomials of degree ≤ n. Let f and g be elements of P_n. Prove that the following function defines an inner product of P_n.

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

Determine the inner product of polynomials

$$f(x) = x^2 + 2x - 1 \text{ and } g(x) = 4x + 1$$

Solution

Axiom 1: $\langle f, g\rangle = \int_0^1 f(x)g(x)dx = \int_0^1 g(x)f(x)dx = \langle g, f\rangle$

Axiom 2: $\langle f+g, h\rangle = \int_0^1 [f(x)+g(x)]h(x)dx$

$$= \int_0^1 [f(x)h(x)+g(x)h(x)]dx$$

$$= \int_0^1 [f(x)h(x)]dx + \int_0^1 g(x)h(x)dx$$

$$= \langle f, h\rangle + \langle g, h\rangle$$

We now find the inner product of the functions $f(x) = x^2 + 2x - 1$ and $g(x) = 4x + 1$

$$\left\langle x^2 + 2x - 1, 4x + 1 \right\rangle = \int_0^1 (x^2 + 2x - 1)(4x + 1)\,dx$$

$$= \int_0^1 (4x^3 + 9x^2 - 2x - 1)\,dx$$

$$= 2$$

Norm of a Vector

The norm of a vector in $\mathbf{R}^n$ can be expressed in terms of the dot product as follows

$$\|(x_1, x_2, \cdots, x_n)\| = \sqrt{(x_1^2 + \cdots + x_n^2)}$$
$$= \sqrt{(x_1, x_2, \cdots, x_n)\cdot(x_1, x_2, \cdots, x_n)}$$

Generalize this definition:
The norms in general vector space do not necessary have geometric interpretations, but are often important in numerical work.

Definition

Let V be an inner product space. The **norm** of a vector **v** is denoted $\|\mathbf{v}\|$ and it defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Example 4

Consider the vector space P_n of polynomials with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

The norm of the function *f* **generated** by this inner product is

$$\| f \| = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 [f(x)]^2 dx}$$

Determine the norm of the function $f(x) = 5x^2 + 1$.

Solution Using the above definition of norm, we get

$$\begin{aligned} \|5x^2+1\| &= \sqrt{\int_0^1 [5x^2+1]^2 dx} \\ &= \sqrt{\int_0^1 [25x^4+10x^2+1]dx} \\ &= \sqrt{\frac{28}{3}} \end{aligned}$$

The norm of the function $f(x) = 5x^2 + 1$ is $\sqrt{\frac{28}{3}}$.

Example 2'

Consider the vector space M_{22} of 2×2 matrices. Let **u** and **v** defined as follows be arbitrary 2×2 matrices.

$$\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

It is known that the function $\langle \mathbf{u}, \mathbf{v} \rangle = ae + bf + cg + dh$ is an inner product on M_{22} by Example 2.

The norm of the matrix is $\mathbf{u} = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{a^2 + b^2 + c^2 + d^2}$

Angle between two vectors

The dot product in $\mathbf{R}^n$ was used to define angle between vectors. The angle θ between vectors **u** and **v** in $\mathbf{R}^n$ is defined by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Definition

Let V be an inner product space. The **angle** θ between two nonzero vectors **u** and **v** in V is given by

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example 5

Consider the inner product space P_n of polynomials with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

The angle between two nonzero functions *f* and *g* is given by

$$\cos\theta = \frac{\langle f, g \rangle}{\| f \| \| g \|} = \frac{\int_0^1 f(x)g(x)dx}{\| f \| \| g \|}$$

Determine the cosine of the angle between the functions

$$f(x) = 5x^2 \text{ and } g(x) = 3x$$

Solution We first compute $\|f\|$ and $\|g\|$.

$$\left\|5x^2\right\| = \sqrt{\int_0^1 [5x^2]^2 dx} = \sqrt{5} \text{ and } \left\|3x\right\| = \sqrt{\int_0^1 [3x]^2 dx} = \sqrt{3}$$

Thus

$$\cos\theta = \frac{\int_0^1 f(x)g(x)dx}{\| f \| \| g \|} = \frac{\int_0^1 (5x^2)(3x)dx}{\sqrt{5}\sqrt{3}} = \frac{\sqrt{15}}{4}$$

Example 2"

Consider the vector space M_{22} of 2×2 matrices. Let **u** and **v** defined as follows be arbitrary 2×2 matrices.

$$\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

It is known that the function $\langle \mathbf{u}, \mathbf{v} \rangle = ae + bf + cg + dh$ is an inner product on M_{22} by Example 2.

The norm of the matrix is $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{a^2 + b^2 + c^2 + d^2}$

The angle between **u** and **v** is

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{ae + bf + cg + dh}{\sqrt{a^2 + b^2 + c^2 + d^2}\sqrt{e^2 + f^2 + g^2 + h^2}}$$

Orthogonal Vectors

Def. Let V be an inner product space. Two nonzero vectors **u** and **v** in V are said to be **orthogonal** if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

Example 6

Show that the functions $f(x) = 3x - 2$ and $g(x) = x$ are orthogonal in P_n with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Solution

$$\langle 3x - 2, x \rangle = \int_0^1 (3x - 2)(x)dx = [x^3 - x^2]_0^1 = 0$$

Thus the functions f and g are orthogonal in this inner product Space.

Distance

As for norm, the concept of distance will not have direct geometrical interpretation. It is however, useful in numerical mathematics to be able to discuss how far apart various functions are.

Definition

Let V be an inner product space with vector norm defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

The **distance** between two vectors (points) **u** and **v** is defined $d(\mathbf{u},\mathbf{v})$ and is defined by

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| \quad (= \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v} \rangle})$$

Example 7

Consider the inner product space P_n of polynomials discussed earlier. Determine which of the functions $g(x) = x^2 - 3x + 5$ or $h(x) = x^2 + 4$ is closed to $f(x) = x^2$.

Solution

$$[d(f,g)]^2 = \langle f-g, f-g \rangle = \langle 3x-5, 3x-5 \rangle = \int_0^1 (3x-5)^2\, dx = 13$$

$$[d(f,h)]^2 = \langle f-h, f-h \rangle = \langle -4, -4 \rangle = \int_0^1 (-4)^2\, dx = 16$$

Thus $d(f,g) = \sqrt{13}$ and $d(f,h) = 4$.

The distance between f and h is 4, as we might suspect, g is closer than h to f.

Inner Product on C^n

For a complex vector space, the first axiom of inner product is modified to read $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$. An inner product can then be used to define norm, orthogonality, and distance, as far a real vector space.

Let $\mathbf{u} = (x_1, \ldots, x_n)$ and $\mathbf{v} = (y_1, \ldots, y_n)$ be element of $\mathbf{C}^n$. The most useful inner product for $\mathbf{C}^n$ is

$$\langle \mathbf{u}, \mathbf{v} \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$$

$\Rightarrow$

※ $\mathbf{u} \perp \mathbf{v}$ if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

※ $\|\mathbf{u}\| = \sqrt{x_1 \bar{x}_1 + \cdots x_n \bar{x}_n}$

※ $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

Example 8

Consider the vectors **u** = (2 + 3*i*, –1 + 5*i*), v = (1 + *i*, – *i*) in $\mathbf{C}^2$.

Compute

(a) 〈 **u**, **v** 〉 , and show that u and v are orthogonal.

(b) ||**u**|| and ||**v**||

(c) *d*(**u**, **v**)

Solution

$$(a)\ \langle \mathbf{u}, \mathbf{v} \rangle = (2+3i)(1-i)+(-1+5i)(i) = 5+i-i-5 = 0$$

thus **u** and **v** are orthogonal.

$$(b)\ \|\mathbf{u}\| = \sqrt{(2+3i)(2-3i)+(-1+5i)(-1-5i)} = \sqrt{13+26} = \sqrt{39}$$

$$\|\mathbf{v}\| = \sqrt{(1+i)(1-i)+(-i)(i)} = \sqrt{3}$$

$$(c)\ d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(2+3i, -1+5i)-(1+i, -i)\|$$

$$= \|(1+2i, -1+6i)\|$$

$$= \sqrt{(1+2u)(1-2i)+(-1+6i)(-1-6i)} = \sqrt{5+37} = \sqrt{42}$$