University of Anbar
College of Engineering
Dept. of Electrical Engineering

Calculus_III (2st Stage)
Abdullah Fawzi Shafeeq
Lecture Four

List of Content for Lecture Four

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Lines and Planes in Space

4.1. Lines and Line Segments in Space

In the plane, a line is determined by a **point** and a number giving the **slope** of the line. In space, a line is determined by a **point** and a **vector** giving the direction of the line.
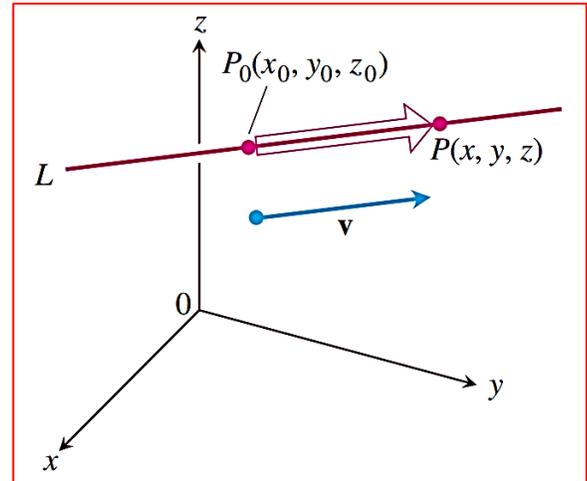
$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \tag{1}$$

If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is
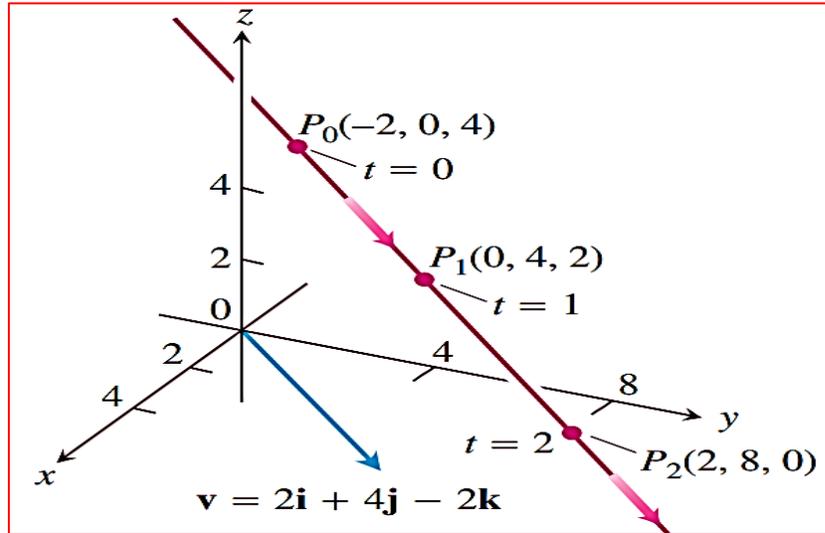
$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

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Ex. Find parametric equations for the line through $(-2, 0, 4)$ parallel to $v = 2i + 4j - 2k$

Sol. With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1 i + v_2 j + v_3 k$ equal to $2i + 4j - 2k$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t. \qquad \blacksquare$$

Ex. Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Sol. The vector

$$\vec{PQ} = (1 - (-3))i + (-1 - 2)j + (4 - (-3))k$$
$$= 4i - 3j + 7k$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t. $\blacksquare$

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The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector **v**. Rewriting Equation (2), we have

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}. \tag{4}$$

Initial   Time   Speed   Direction
position

Ex.   A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Sol.   We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\text{speed})\mathbf{u}$$

$$= 0 + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$$

$$= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

When $t = 10$ sec,

$$\mathbf{r}(10) = 200\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$= \left\langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \right\rangle.$$

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4.1.1. The Distance from a Point to a Line in Space

To find the distance from a point **S** to a line that passes through a point **P** parallel to a vector $\vec{v}$, we find the absolute value of the scalar component $\overrightarrow{PS}$ of in the direction of a vector normal to the line . In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}| \sin \theta$.
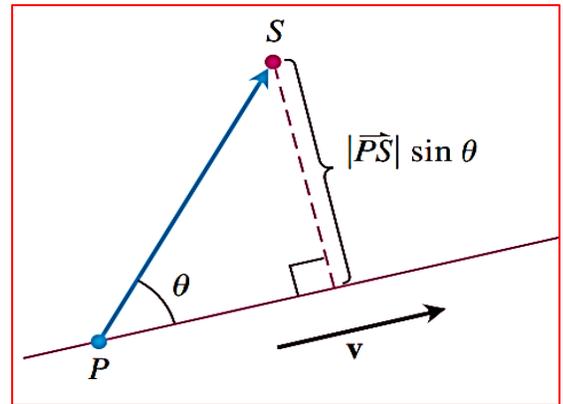
Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

Ex. Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Sol. We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

H.W: Find the distance from the point $S(0,0,12)$ to the line
$$L: x = 4t \quad , y = -2t \quad , z = 2t$$

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4.2. Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane.
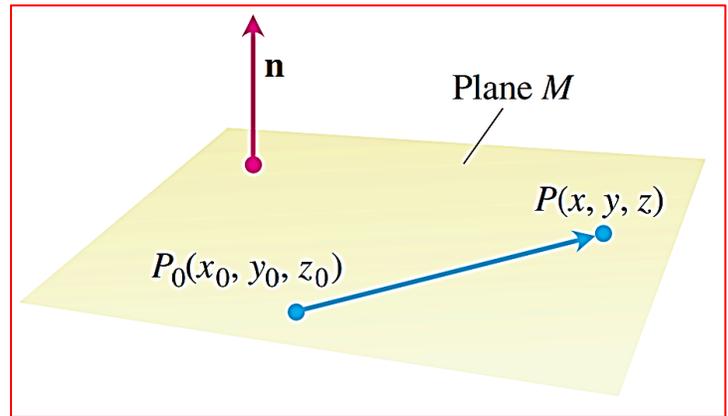
Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ (Figure   ). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D,$ where

$D = Ax_0 + By_0 + Cz_0$

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Ex. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Sol. The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22. \qquad \blacksquare$$

Ex. Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Sol. We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6. \qquad \blacksquare$$

4.3. Important Notes of Planes and Lines

1. We can find the Lines equation (or parametric equation), if we have **parallel vector** and any **point** on the **lines**.

2. If the line has **two point**, we find the vector from these point which represente the parallel vector and use the **initial point** to determined the parametric equation of line.

3. We can find the equation of **plane** if we have perpendicular or **normal vector** and **point** lies on it.

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4. To find any **point** on the plane (or from plane equation), set $y = z = 0$ to find $P(x_0, 0, 0)$.

5. To find the **sharing point** of two planes, set $z = 0$ and solve them equation to find x_0 and y_0 to get $P(x_0, y_0, 0)$.

6. To find the **intersection point** of **line** and **plane**, set the parametric equation of line in the plane equation to find the **parameter (t)** . After that, apply (t) in the parametric equation of line to find $P(x_0, y_0, z_0)$.

7. If we have two planes (plane1 with $\overrightarrow{n_1}$) and (plane2 with $\overrightarrow{n_2}$), we say
   - plane1 $||$ plane2 if $\overrightarrow{n_1} \times \overrightarrow{n_2} = 0$
   - plane1 $\perp$ plane2 if $\overrightarrow{n_1}.\overrightarrow{n_2} = 0$

8. To find **angle θ** between **two planes** we use this formula

$$\overrightarrow{n_1}.\overrightarrow{n_2} = |\overrightarrow{n_1}||\overrightarrow{n_2}| \cos\theta \implies \theta = \cos^{-1}\left(\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}\right)$$

9. To find the distance between **plane** $(Ax + By + Cz = D)$ and **point** $S(x_1, y_1, z_1)$.

$$Distance = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}$$

   - If Distance is **+ve**, that's mean the point is located **upper** to the plane.
   - If Distance is **–ve**, that's mean the point is located **under** to the plane.

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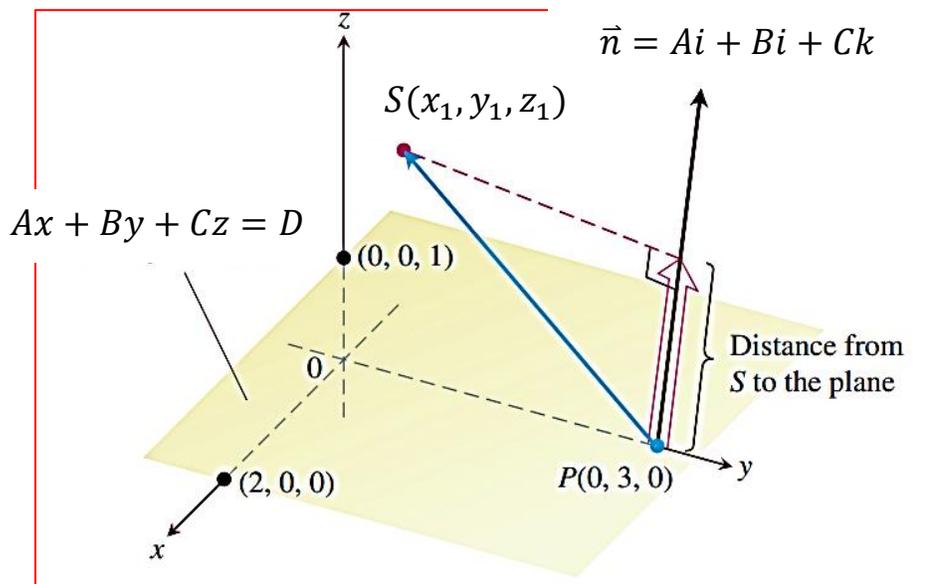
Ex. Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol. The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ (Figure   ) and therefore parallel to $\mathbf{n_1} \times \mathbf{n_2}$. Turning this around, $\mathbf{n_1} \times \mathbf{n_2}$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n_1} \times \mathbf{n_2}$ will do as well. ∎
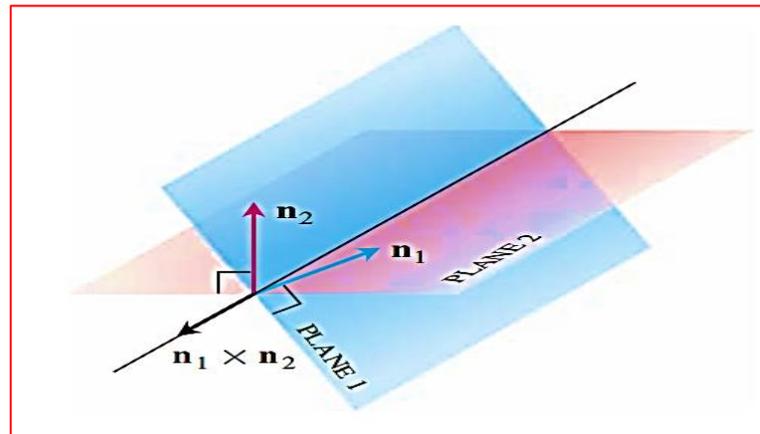
Ex. Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Sol. We find a vector parallel to the line and a point on the line and use Equations (3). Prev. Ex. identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z. The different choices would simply give different parametrizations of the same line. ∎

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Ex. Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Sol. The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

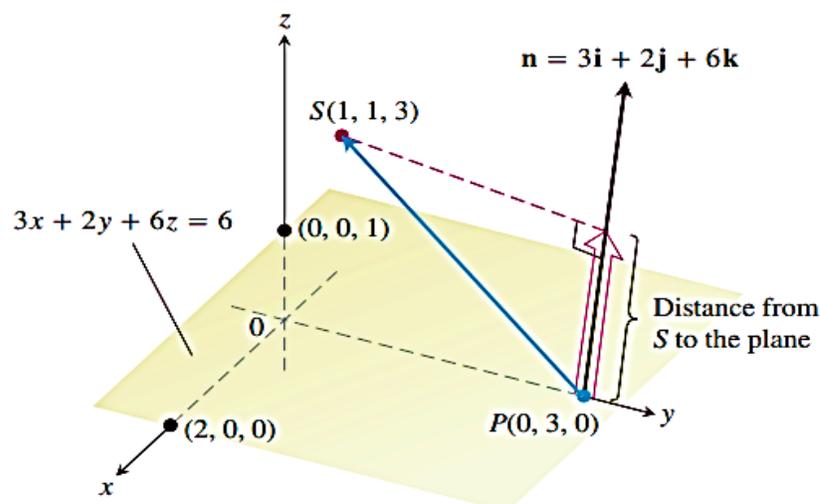$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1.$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

Ex. Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Sol. We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector **n** normal to the plane (Figure). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

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The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept $(0, 3, 0)$, then

$$\vec{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k}$$
$$= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$
$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

The distance from S to the plane is

$$d = \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \qquad \text{length of } \text{proj}_{\mathbf{n}} \vec{PS}$$

$$= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right|$$

$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}. \qquad ■$$

This example can be solved by one-step by using the **Note. 9.**

$$Distance = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}$$

$$Distance = \frac{3(1) + 2(1) + 6(3) - 6}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{17}{7} \text{ the point is located upper the plane.}$$

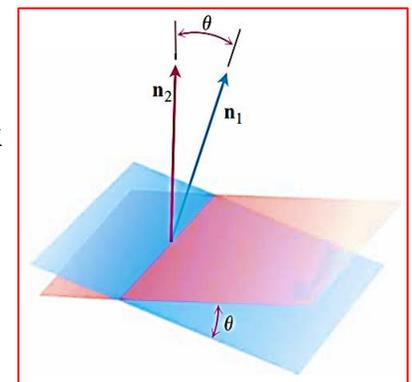**Ex.** Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Sol. The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right)$$

$$= \cos^{-1} \left(\frac{4}{21} \right)$$

$$\approx 1.38 \text{ radians.} \qquad \text{About 79 deg} \qquad ■$$

Note: Exercises 12.5. in Thomas Calculus 12ᵗʰ edition have the similar problems above.