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Lecture Three

Product of Vectors

We have two kinds of vector multiplication, they are:

3.1. Dot Product (Scalar Product)

DEFINITION The dot product $\mathbf{u} \cdot \mathbf{v}$ (“ \mathbf{u} dot \mathbf{v} ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Ex.

$$\begin{aligned} \text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7 \end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3\mathbf{j} + \mathbf{k} \right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

The Dot product is defined by this formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$$

Let $\mathbf{u} = x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k}$

$\mathbf{v} = x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k}$

We know the angle θ between \mathbf{i}, \mathbf{j} , and \mathbf{k} is 90°

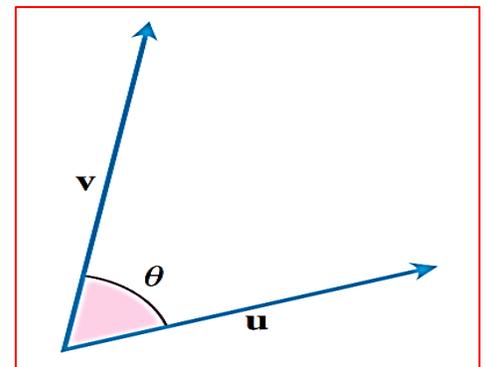
the angle θ between (\mathbf{i}, \mathbf{i}) , (\mathbf{j}, \mathbf{j}) and (\mathbf{k}, \mathbf{k}) is 0° so

$$\mathbf{u} \cdot \mathbf{v} = (x_1 \mathbf{i} + y_1 \mathbf{j} + z_1 \mathbf{k}) \cdot (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k})$$

$$= x_1 \mathbf{i} \cdot (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k}) + y_1 \mathbf{j} \cdot (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k}) + z_1 \mathbf{k} \cdot (x_2 \mathbf{i} + y_2 \mathbf{j} + z_2 \mathbf{k})$$

but $\mathbf{i} \cdot \mathbf{i} = 1 * 1 * \cos 0 = 1$ and $\mathbf{i} \cdot \mathbf{j} = 1 * 1 * \cos 90^\circ = \text{zero}$

Therefore $\mathbf{u} \cdot \mathbf{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$





THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right). \quad \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right)$$

Ex. Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Sol. We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right) = \cos^{-1} \left(\frac{-4}{(3)(7)} \right) \approx 1.76 \text{ radians.}$$

The angle formula applies to two-dimensional vectors as well.

Ex. Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$

Sol. The angle θ is the angle between the vectors \vec{CA} and \vec{CB} . The component forms of these two vectors are

$$\vec{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \vec{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

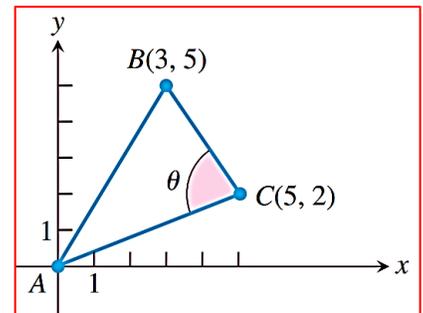
$$\vec{CA} \cdot \vec{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\vec{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\vec{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}| |\vec{CB}|} \right) \\ &= \cos^{-1} \left(\frac{4}{(\sqrt{29})(\sqrt{13})} \right) \\ &\approx 78.1^\circ \quad \text{or} \quad 1.36 \text{ radians.} \end{aligned}$$





3.1.1. Perpendicular (Orthogonal) Vectors

Two nonzero vectors \mathbf{u} and \mathbf{v} are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$ or 90°

DEFINITION Vectors \mathbf{u} and \mathbf{v} are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Ex. To determine if two vectors are orthogonal, calculate their dot product.

- (a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.
- (b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.
- (c) $\mathbf{0}$ is orthogonal to every vector \mathbf{u} since

$$\begin{aligned}\mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0.\end{aligned}$$

3.1.2. Properties of the Dot Product

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
- $\mathbf{0} \cdot \mathbf{u} = 0$.

3.1.3. Vector Projection

The vector projection of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ is the vector \overrightarrow{PR} determined by dropping a perpendicular from Q to the line PS . The notation for this vector is

$\text{proj}_v \mathbf{u}$ “the vector projection of \mathbf{u} onto \mathbf{v} ”

If \mathbf{u} represents a force, then $\text{proj}_v \mathbf{u}$ represents the effective force in the direction of \mathbf{v}

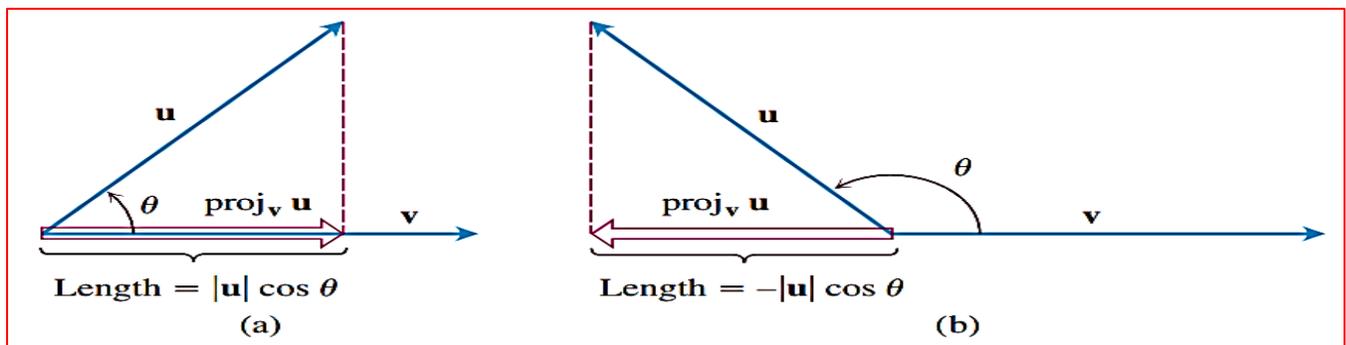
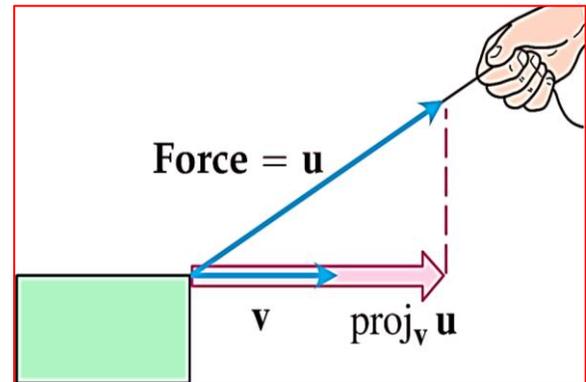
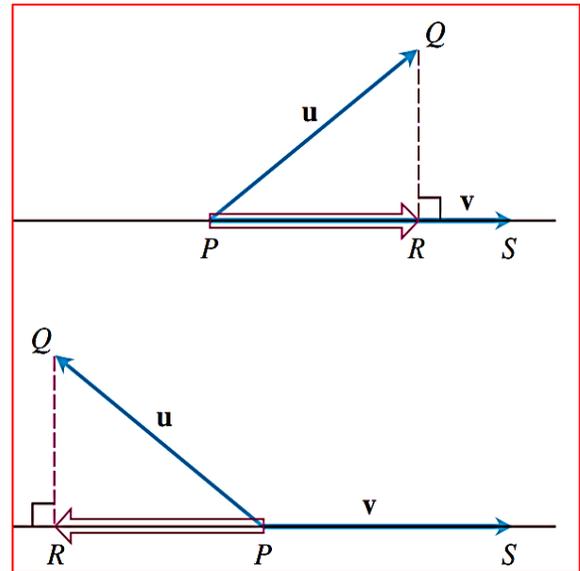
If the angle θ between \mathbf{u} and \mathbf{v} is acute, $\text{proj}_v \mathbf{u}$ has length $|\mathbf{u}| \cos \theta$ and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$. If θ is obtuse, $\cos \theta < 0$ and $\text{proj}_v \mathbf{u}$ has length $-|\mathbf{u}| \cos \theta$ and direction $\frac{-\mathbf{v}}{|\mathbf{v}|}$. In both cases

$$\text{proj}_v \mathbf{u} = (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$|\mathbf{u}| \cos \theta = \frac{|\mathbf{u}| |\mathbf{v}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$$





The number $|u| \cos \theta$ is called the **scalar component** of \mathbf{u} in the direction of \mathbf{v} (or of \mathbf{u} onto \mathbf{v}). To summarize

The vector projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}. \quad (1)$$

The scalar component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (2)$$

Ex. Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

Sol. We find $\text{proj}_{\mathbf{v}} \mathbf{u}$ from Equation (1):

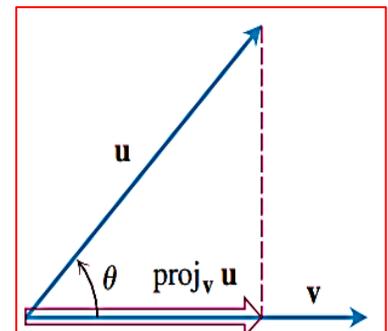
$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}. \end{aligned}$$

We find the scalar component of \mathbf{u} in the direction of \mathbf{v} from Equation (2):

$$\begin{aligned} |\mathbf{u}| \cos \theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}. \end{aligned}$$

To express \mathbf{u} as a sum of orthogonal vectors.

$$\mathbf{u} = \underbrace{\text{proj}_{\mathbf{v}} \mathbf{u}}_{\text{Parallel to } \mathbf{v}} + \underbrace{(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u})}_{\text{Orthogonal to } \mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} + \left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right)$$



Ex. Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of \mathbf{F} in the direction of \mathbf{v} .

Sol. The vector projection is

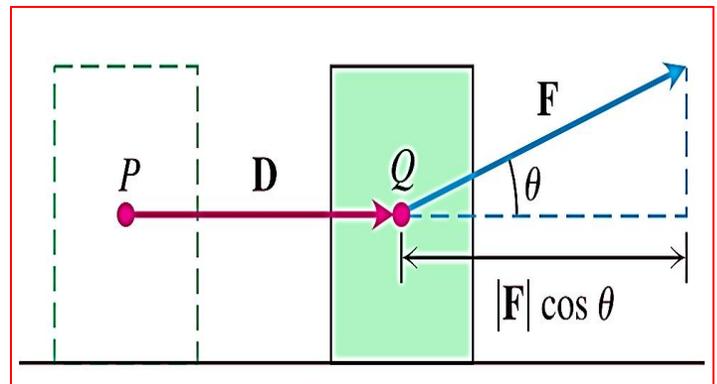
$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \\ &= \frac{5 - 6}{1 + 9} (\mathbf{i} - 3\mathbf{j}) = -\frac{1}{10} (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10} \mathbf{i} + \frac{3}{10} \mathbf{j}. \end{aligned}$$

The scalar component of \mathbf{F} in the direction of \mathbf{v} is

$$|\mathbf{F}| \cos \theta = \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{5 - 6}{\sqrt{1 + 9}} = -\frac{1}{\sqrt{10}}.$$

3.1.4. Work done

If a force \mathbf{F} moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of \mathbf{F} in the direction of \mathbf{D} . If θ is the angle between \mathbf{F} and \mathbf{D} , then



(scalar component of \mathbf{F}
 in the direction of \mathbf{D}) (length of \mathbf{D})

$$(|\mathbf{F}| \cos \theta) |\mathbf{D}|$$

$$\mathbf{F} \cdot \mathbf{D}.$$

DEFINITION The work done by a constant force \mathbf{F} acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

Ex. If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60^\circ$, the work done by \mathbf{F} in acting from P to Q is

Sol.

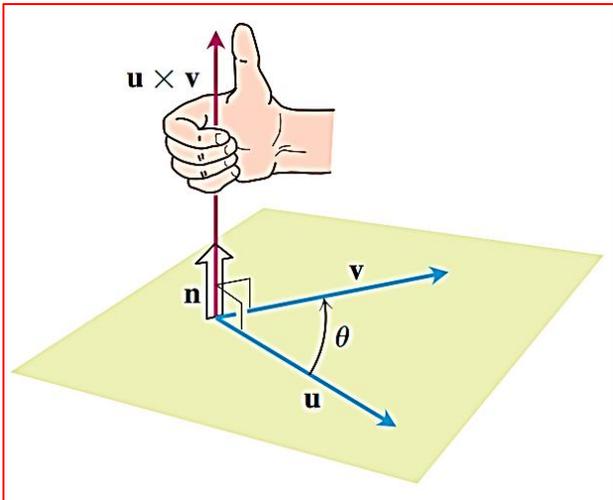
$$\begin{aligned} \text{Work} &= \mathbf{F} \cdot \mathbf{D} && \text{Definition} \\ &= |\mathbf{F}| |\mathbf{D}| \cos \theta \\ &= (40)(3) \cos 60^\circ && \text{Given values} \\ &= (120)(1/2) = 60 \text{ J (joules)}. \end{aligned}$$

Note: Exercises 12.3. in Thomas Calculus 12th edition have the similar problems above.

3.2. The Cross Product

The cross product of two non zero vectors \vec{u} and \vec{v} inclined to each other at angle θ (θ is counted +ve in the anti-clockwise direction and -ve in the clockwise direction) is a vector quantity denoted by $\vec{u} \times \vec{v}$ and defined as follows

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}| |\mathbf{v}| \sin \theta) \mathbf{n}$$



3.2.1. Parallel Vector

Since the sines ($\sin \theta$) of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\mathbf{0}$. If one or both of \vec{u} and \vec{v} are zero, we also define $\vec{u} \times \vec{v}$ to be zero. This way, the cross product of two vectors \vec{u} and \vec{v} is zero if and only if \vec{u} and \vec{v} are parallel or one or both of them are **zero**.

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

3.2.2. Properties of the Cross Product

The cross product obeys the following laws.

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

$\mathbf{k} = \mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i})$

$\mathbf{j} = \mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k})$

$\mathbf{i} = \mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j})$

$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$

$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$

$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$

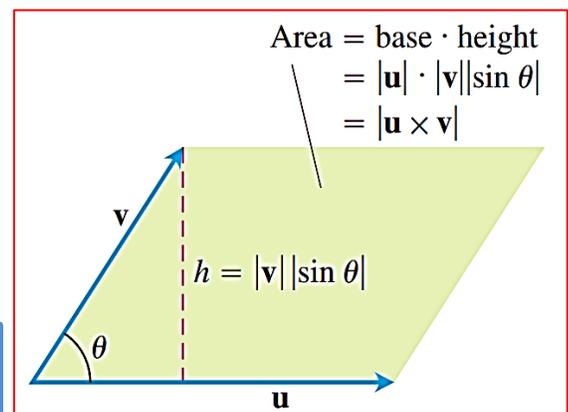
$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$

Diagram for recalling these products

3.2.3. Area of Parallelogram

The area of the parallelogram is determined by magnitude of $|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|$ because $\vec{\mathbf{n}}$ is a unit vector, $|\vec{\mathbf{u}}|$ being the base of the parallelogram and $|\vec{\mathbf{v}}| \sin \theta$ the height as shown in figure.

$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$



Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Ex. Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Sol.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \quad \blacksquare$$

Ex. Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ As shown in figure

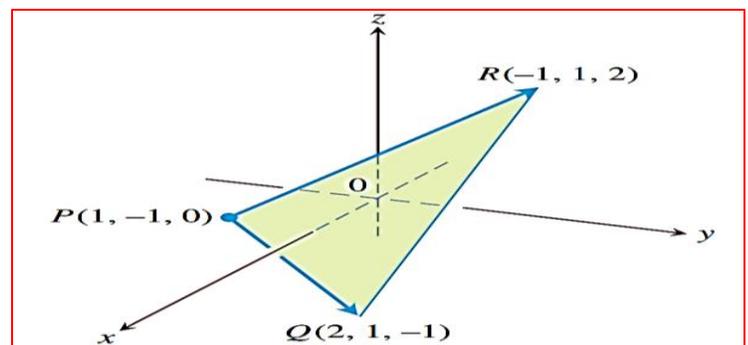
Sol. The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$



Ex. Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Sol. The area of the parallelogram determined by P , Q , and R is

$$\begin{aligned} |\vec{PQ} \times \vec{PR}| &= |6\mathbf{i} + 6\mathbf{k}| && \text{Values from previous Example} \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}. \end{aligned}$$

The triangle's area is half of this, or $3\sqrt{2}$. ■

Ex. Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Sol. Since $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane, its direction \mathbf{n} is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}. \quad \blacksquare$$

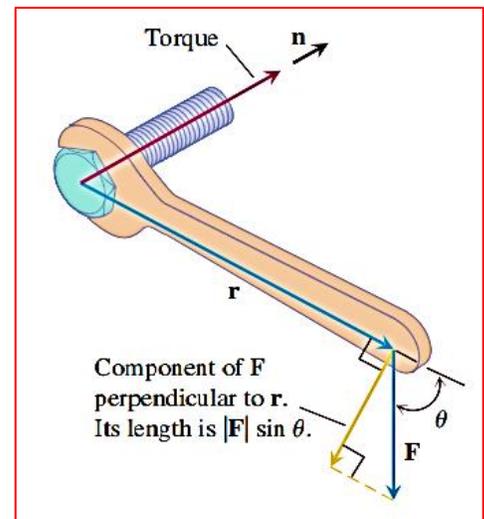
3.2.4. Torque

When we turn a bolt by applying a force \mathbf{F} to a wrench, we produce a torque that causes the bolt to rotate.

Magnitude of torque vector = $|\mathbf{r}| |\mathbf{F}| \sin \theta$

Or $|\vec{\mathbf{r}} \times \vec{\mathbf{F}}|$. If we let \mathbf{n} be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is or $\vec{\mathbf{r}} \times \vec{\mathbf{F}}$

Torque vector = $(|\mathbf{r}| |\mathbf{F}| \sin \theta) \mathbf{n}$.

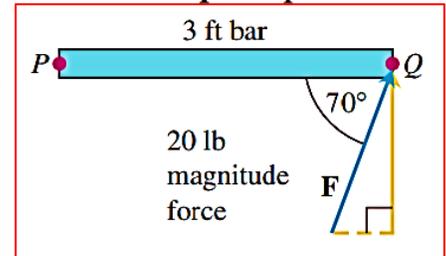


Recall that we defined $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ to be 0 when $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. This is consistent with the torque interpretation as well. If the force \mathbf{F} is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

Ex. The magnitude of the torque generated by force **F** at the pivot point **P**

Sol.

$$\begin{aligned}
 |\vec{PQ} \times \mathbf{F}| &= |\vec{PQ}| |\mathbf{F}| \sin 70^\circ \\
 &\approx (3)(20)(0.94) \\
 &\approx 56.4 \text{ ft-lb.}
 \end{aligned}$$



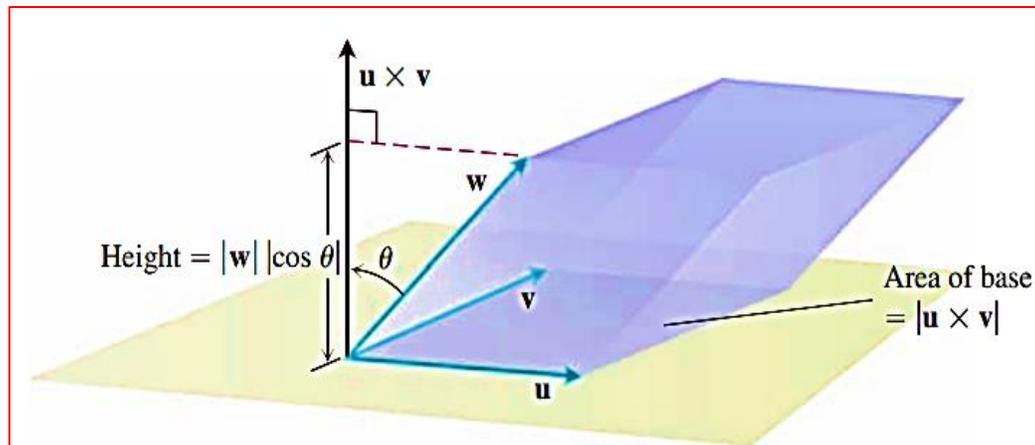
In this example the torque vector is pointing out of the page toward you.

3.2.5. Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w** (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by **u**, **v**, and **w** (Figure 12.34). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base



$$\begin{aligned}
 \text{Volume} &= \text{area of base} \cdot \text{height} \\
 &= |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta| \\
 &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|
 \end{aligned}$$

Note: The **dot** and **cross** may be interchanged in a triple scalar product without altering its value.

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Ex. Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Sol. Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■

Note: Exercises 12.4. in Thomas Calculus 12th edition have the similar problems above.