Calculus IV Lectures for 2nd class

Lecture Six : Second order Differential Equations
(Non- Homogeneous Equations Type Equations)

Second Order Non-homogeneous Linear Equations

Now, we solve non-homogeneous equations of the form

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=F(x)$$

The procedure has three basic steps. First, we find the homogeneous solution y_h (h stands for "homogeneous") of the ***reduced equation***

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$$

Second, we find a particular solution y_p of the ***complete*** equation. Finally, we add y_p to y_h to form the general solution of the complete equation. So, the final solution is

$$y=y_h+y_p$$

Second-Order Non-homogeneous ODE with constant Coefficients

the general form of 2^{nd} order non-homogeneous ODE with constant Coefficients is:

$$a_2 y'' + a_1 y' + a_0 y = r(x)$$

the Complete Solution is:

$$y(x) = y_h(x) + y_p(x)$$

where: $y_h(x)$ is the homogeneous solution.
$y_p(x)$ is the particular Solution depends on $r(x)$

The are three methods (ways) to find y_p:-

1- the method of undetermined Coefficients.
2- The Inverse operator Method.
3- Variation of Parameters Method

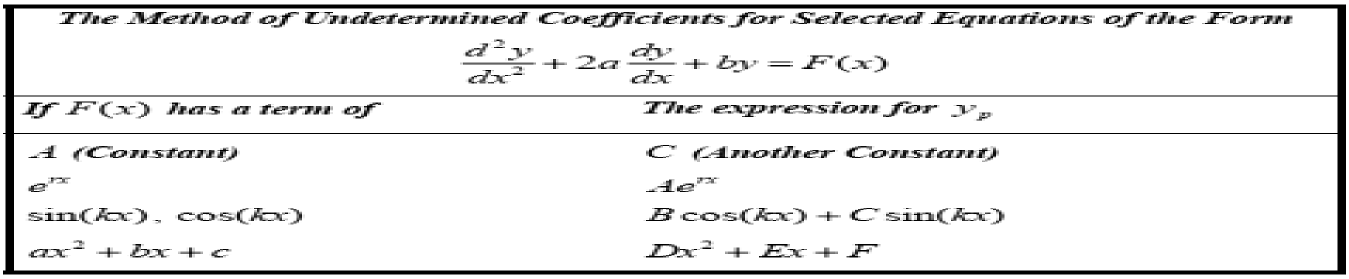
1. the Method of Undetermined Coefficients

This method is used only if $r(x)$ is one of the function in table below.

$r(x)$	y_p
1. k , k = constant	A
2. ke^{ax}	Ae^{ax}
3. $k_n x^n + k_{n-1} x^{n-1} + --- + k_1 x + k_0$	$A_n x^n + A_{n-1} x^{n-1} + ---- A_1 x + A_0$
4. $k \cos bx$ $k \sin bx$ $k_1 \cos bx + k_2 \sin bx$	$\rightarrow A \cos bx + B \sin bx$
5. $k e^{ax} \cos bx$ $k e^{ax} \sin bx$ $e^{ax}(k_1 \cos bx + k_2 \sin bx)$	$\rightarrow e^{ax}(A \cos bx + B \sin bx)$
6. $e^{ax}(k_n x^n + k_{n-1} x^{n-1} + --- k_1 x + k_0)$	$e^{ax}(A_n x^n + A_{n-1} x^{n-1} + --- + A_1 x + A$
7. $(k_n x^n + k_{n-1} x^{n-1} + --- + k_0) \cos bx$ $(k_n x^n + k_{n-1} x^{n-1} + --- + k_0) \sin bx$	$\rightarrow (A_n x^n + A_{n-1} x^n + ---- + A_0) \cos bx$ $+ (B_n x^n + B_{n-1} x^{n-1} + --- B_0) \sin bx$

Undetermined Coefficients

This method gives us the particular solution for selected equations.



The Method of Undetermined Coefficients for Selected Equations of the Form

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$$

If $F(x)$ has a term of	*The expression for y_p*
A (Constant)	C (Another Constant)
e^{rx}	Ae^{rx}
$\sin(kx)$, $\cos(kx)$	$B\cos(kx) + C\sin(kx)$
$ax^2 + bx + c$	$Dx^2 + Ex + F$

Example

Solve the equation $y'' + 3y = e^x$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' + 3y = 0$$

The characteristic equation is

$$D^2 + 3 = 0$$

The roots are $r_1 = j\sqrt{3}$, and $r_2 = -j\sqrt{3} \Rightarrow \alpha = 0$ and $\beta = \sqrt{3}$

So, $$y_h = C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)$$

Since $F(x) = e^x$ then let $y_p = Ae^x \Rightarrow y'_p = Ae^x \Rightarrow y''_p = Ae^x$

Substituting into the differential equation $y'' + 3y = e^x$ we get

$$Ae^x + 3Ae^x = e^x \Rightarrow A + 3A = 1 \Rightarrow A = \frac{1}{4}$$

So, $$y_p = \frac{1}{4}e^x$$

And the complete solution is

$$y = C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x) + \frac{1}{4}e^x$$

Important Note

The expression used for y_p should not have any term similar to the terms of the homogeneous solution. Otherwise, multiply the term that is similar to the homogeneous solution repeatedly by x until it becomes different.

Example

Solve the equation $y'' - 3y' + 2y = 5e^x$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' - 3y' + 2y = 0$$

The characteristic equation is

$$D^2 - 3D + 2 = 0$$

$$(D-1)(D-2) = 0$$

The roots are $r_1 = 1$, and $r_2 = 2$

$$y_h = C_1 e^x + C_2 e^{2x}$$

Since $F(x) = 5e^x$ then let $y_p = Ae^x \Rightarrow \; y_p' = Ae^x \Rightarrow y_p'' = Ae^x$

Substituting into the differential equation $y'' - 3y' + 2y = 5e^x$ we get

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

$$0 = 5e^x \qquad \textit{(Wrong Answer)}$$

The trouble can be traced to the fact that e^x is already a solution in the homogeneous equation $y_h = C_1 e^x + C_2 e^{2x}$.

The appropriate way is to modify the particular solution to replace Ae^x by

$$y_p = Axe^x$$

$$y_p' = Axe^x + Ae^x$$

$$y_p'' = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$$

Substituting into the differential equation $y'' - 3y' + 2y = 5e^x$ we get

$$\left(Axe^x + 2Ae^x\right) - 3\left(Axe^x + Ae^x\right) + 2Axe^x = 5e^x$$

$$-Ae^x = 5e^x$$

$$\Rightarrow \qquad A = -5$$

So, $\qquad y_p = -5xe^x$

The complete solution (general solution) is

$$y = C_1e^x + C_2e^{2x} - 5xe^x$$

Example

Solve the equation

(a) $y'' - 6y' + 9y = e^{3x}$, $\qquad$ (b) $y'' - y' = 5e^x - \sin(2x)$

(c) $y'' - y' - 2y = 4x^3$

Solution

(a) The homogeneous solution y_h can be found using the reduced equation

$$y'' - 6y' + 9y = 0$$

The characteristic equation is

$$D^2 - 6D + 9 = 0$$

$$(D-3)^2 = 0$$

The roots are $\qquad r_1 = r_2 = 3$

$$y_h = (C_1x + C_2)e^{3x}$$

Since $F(x) = e^{3x}$ then let $y_p = Ae^{3x}$. But, Ae^{3x} is similar to the second term of th

homogeneous solution so, let $y_p = Axe^{3x}$. Again Axe^{3x} is also similar to the firs

term of the homogeneous solution. Finally, let

$$y_p = Ax^2e^{3x} \quad \Rightarrow \quad y_p' = 3Ax^2e^{3x} + 2Axe^{3x}$$

$$y_p'' = \left(9Ax^2e^{3x} + 6Axe^{3x}\right) + \left(6Axe^{3x} + 2Ae^{3x}\right)$$
$$= 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

Substituting into the differential equation $y'' - 6y' + 9y = e^{3x}$ we get

$$\left(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}\right) - 6\left(3Ax^2e^{3x} + 2Axe^{3x}\right) + 9Ax^2e^{3x} = e^{3x}$$

$$2Ae^{3x} = e^{3x}$$

$$\Rightarrow \qquad 2A = 1$$

$$\Rightarrow \qquad A = \frac{1}{2}$$

So, $\qquad y_p = \frac{1}{2}x^2e^{3x}$

The general solution is $\qquad y = (C_1x + C_2)e^{3x} + \frac{1}{2}x^2e^{3x}$

(b) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' = 0$$

The characteristic equation is

$$D^2 - D = 0$$
$$D(D-1) = 0$$

The roots are $\qquad r_1 = 1$, and $r_2 = 0$

$$y_h = C_1e^x + C_2$$

Since $F(x) = 5e^x - \sin(2x)$ then let $y_p = Ae^x + B\cos(2x) + C\sin(2x)$. But, Ae^x is similar to the first term of the homogeneous solution so, let

$$y_p = Axe^x + B\cos(2x) + C\sin(2x)$$

$$y_p' = Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)$$

$$y_p'' = Axe^x + Ae^x + Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

$$= Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

Substituting into the differential equation $y'' - y' = 5e^x - \sin(2x)$ we get

$$\left(Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)\right)$$
$$-\left(Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)\right) = 5e^x - \sin(2x)$$

$$Ae^x - (4B + 2C)\cos(2x) + (2B - 4C)\sin(2x) = 5e^x - \sin(2x)$$

$\Rightarrow$ $$A = 5,\ \ (4B + 2C) = 0,\ \ (2B - 4C) = -1$$

or $$A = 5,\quad B = -\frac{1}{10},\quad C = \frac{1}{5}$$

So, $$y_p = 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

The general solution is

$$y = y_h + y_p = C_1e^x + C_2 + 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

(c) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' - 2y = 0$$

The characteristic equation is

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1)=0$$

The roots are $r_1 = 2$, and $r_2 = -1$

$$y_h = C_1 e^{2x} + C_2 e^{-x}$$

Since $F(x) = 4x^3$ then let

$$y_p = Ax^3 + Bx^2 + Cx + D \quad \Rightarrow \quad y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

Substituting into the differential equation $y'' - y' - 2y = 4x^3$ we get

$$6Ax + 2B - (3Ax^2 + 2Bx + C) - 2(Ax^3 + Bx^2 + Cx + D) = 4x^3$$

$$-2Ax^3 - (3A+2B)x^2 + (6A-2B-2C)x + (2B-C-2D) = 4x^3$$

$$\Rightarrow \quad A = -2$$

$$3A + 2B = 0 \quad \Rightarrow \quad 3(-2) + 2B = 0 \quad \Rightarrow \quad B = 3$$

$$6A - 2B - 2C = 0 \quad \Rightarrow \quad 6(-2) - 2(3) - 2C = 0 \quad \Rightarrow \quad C = -9$$

$$2B - C - 2D = 0 \quad \Rightarrow \quad 2(3) - (-9) - 2D = 0 \quad \Rightarrow \quad D = \frac{15}{2}$$

So, $y_p = -2x^3 + 3x^2 - 9x + 7.5$

The general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} - 2x^3 + 3x^2 - 9x + 7.5$$

Example

- $y'' = 9x^2 + 2x - 1$

$D^2 = 0 \Rightarrow r_1 = r_2 = 0 \Rightarrow y_h = C_1 x + C_2$

$y_p = x^2\left(Ax^2 + Bx + C\right)$

- $y'' - y' = x$

$D^2 - D = 0$

$D(D-1) = 0 \Rightarrow r_1 = 0$ and $r_2 = 1 \Rightarrow y_h = C_1 + C_2 e^x$

$y_p = x(Ax + B)$

- $y'' - 5y = 3e^x - 2x + 1$

$D^2 - 5 = 0$

$(D-\sqrt{5})(D+\sqrt{5}) = 0 \Rightarrow r_1 = \sqrt{5}$ and $r_2 = -\sqrt{5}$

$y_h = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x}$

$y_p = Ae^x + Bx + C$

- $y'' - 4y' + 3y = e^{3x} + 2$

$D^2 - 4D + 3 = 0$

$(D-3)(D-1) = 0 \Rightarrow r_1 = 3$ and $r_2 = 1 \Rightarrow y_h = C_1 e^{3x} + C_2 e^x$

$y_p = Axe^{3x} + B$

- $y'' + y = 6e^{x} + 6\cos(x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1\cos(x) + C_2\sin(x)$

  $y_p = Ae^{3x} + x\left(B\cos(x) + C\sin(x)\right)$

- $y'' - 2y' + y = xe^{x}$

  $D^2 - 2D + 1 = 0$

  $(D-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Rightarrow y_h = \left(C_1x + C_2\right)e^{x}$

  $y_p = (Ax + B)(x^2e^{x})$

- $y'' + y = x^2\sin(2x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1\cos(x) + C_2\sin(x)$

  $y_p = \left(Ax^2 + Bx + C\right)\ (\cos(2x) + \sin(2x))$

Notes:

To find the roots of an equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$

- r is a root of $f(x)$ if $f(r) = 0$.
- r is a repeated root of $f(x)$ if $f'(r) = 0$.
- If r is a root then r must be a factor of a_n.
- If r is a root then $f(x)$ is divided by $(x - r)$.

Example

- $x^3+4x^2-3x-18=0$

Factors of 18 are: $(\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \quad \pm 9, \quad \pm 18)$

$$f(1)=(1)^3+4(1)^2-3(1)-18=-16\neq 0$$

$$f(-1)=(-1)^3+4(-1)^2-3(-1)-18=-12\neq 0$$

$$f(2)=(2)^3+4(2)^2-3(2)-18=0 \qquad \Rightarrow \qquad r_1=2\,.$$

$$f'(x)=3x^2+8x-3$$

$$f'(2)=3(2)^2+8(2)-3=25\neq 0 \qquad \Rightarrow \qquad r_1=2 \text{ is not a repeated root.}$$

$$\begin{array}{r|rrrr}
 & x^2 & +6x & +9 & \\ \cline{2-5}
x \quad -2 & x^3 & +4x^2 & -3x & -18 \\ \hline
 & \mp x^3 & \pm 2x^2 & & \\ \cline{2-4}
 & & 6x^2 & -3x & \\
 & & \mp 6x^2 & \pm 12x & \\ \cline{3-5}
 & & & 9x & -18 \\
 & & & \mp 9x & \pm 18 \\ \cline{4-5}
 & & & 0 & 0
\end{array}$$

$$x^2+6x+9=0 \qquad \Rightarrow \qquad (x+3)^2=0 \qquad \Rightarrow \qquad r_2=r_3=-3\,.$$

Ex :- Solve $y'' - y' - 2y = 4x^2$

Sol :- $\lambda^2 - \lambda - 2 = 0 \Rightarrow \lambda_1 = -1 , \lambda_2 = 2$

$y_h = c_1 e^{-x} + c_2 e^{2x}$

$$\left.\begin{array}{l} y_p = A_2 x^2 + A_1 x + A_0 \\ y_p' = 2A_2 x + A_1 \\ y_p'' = 2A_2 \end{array}\right\} \rightarrow \text{Substitute in the given ODE}$$

$(2A_2) - (2A_2 x + A_1) - 2(A_2 x^2 + A_1 x + A_0) = 4x^2$

$(-2A_2) x^2 + (-2A_2 - 2A_1) x + (2A_2 - A_1 - 2A_0) = 4x^2$

$-2A_2 = 4 \Rightarrow \boxed{A_2 = -2}$

$-2A_2 - 2A_1 = 0 \Rightarrow A_1 = -A_2 \Rightarrow \boxed{A_1 = 2}$

$2A_2 - A_1 - 2A_0 = 0 \Rightarrow \boxed{A_0 = -3}$

$\therefore y_p = -2x^2 + 2x - 3$

$y = y_h + y_p = c_1 e^{-x} + c_2 e^{2x} - 2x^2 + 2x - 3$

Ex :- Solve $y'' - y' - 2y = e^{3x}$

Sol :- $y_h = c_1 e^{-x} + c_2 e^{2x}$

$y_p = A e^{3x} \Rightarrow y_p' = 3A e^{3x} \Rightarrow y_p'' = 9A e^{3x}$

$\therefore 9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \Rightarrow 4Ae^{3x} = e^{3x}$

$\therefore 4A = 1 \Rightarrow \boxed{A = \frac{1}{4}}$

$\therefore y_p = \frac{1}{4} e^{3x}$ and $y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}$

Ex :- Solve $y'' + y = 12 \cos^2 x$

Sol : $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = c_1 \cos x + c_2 \sin x$

$r(x) = 12 \cos^2 x = 12 \frac{1 + \cos 2x}{2} = 6 + 6 \cos 2x$

$\therefore y_p = C + A \cos 2x + B \sin 2x$

$y'_p = -2A\sin 2x + 2B\cos 2x$

$y''_p = -4A\cos 2x - 4B\sin 2x$

$\therefore -4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x + C = 6 + 6\cos 2x$

$-3A\cos 2x - 3B\sin 2x + C = 6 + 6\cos 2x$

$\therefore \boxed{C=6} \;,\; \boxed{A=-2} \;,\; \boxed{B=0}$

$\therefore y_p = 6 - 2\cos 2x$

$y = c_1\cos x + c_2\sin x + 6 - 2\cos 2x$

Ex:- Solve $y'' - 4y' + 4y = 4e^{2x}$

Sol:- $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \Rightarrow y_h = c_1 e^{2x} + c_2 x e^{2x}$

$y_p = A x^2 e^{2x}$

$y'_p = 2A x^2 e^{2x} + 2Ax e^{2x}$

$y''_p = 4Ax^2 e^{2x} + 4Axe^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$= 4Ax^2 e^{2x} + 8Axe^{2x} + 2Ae^{2x}$

$\therefore \cancel{4Ax^2 e^{2x}} + \cancel{8Axe^{2x}} + 2Ae^{2x} - \cancel{8Ax^2 e^{2x}} - \cancel{8Axe^{2x}} + \cancel{4Ax^2 e^{2x}} = 4e^{2}$

$2A = 4 \Rightarrow \boxed{A=2}$

$\therefore y_p = 2x^2 e^{2x}$ and $y = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 e^{2x}$

Ex: Solve $y'' - 9y = e^{3x} + \sin 3x$

Sol:- $y_h = c_1 e^{-3x} + c_2 e^{3x}$

$y_p = Cx e^{3x} + A\cos 3x + B\sin 3x$

$y'_p = 3Cxe^{3x} + Ce^{3x} - 3A\sin 3x + 3B\cos 3x$

$y''_p = 9Cxe^{3x} + 3Ce^{3x} + 3Ce^{3x} - 9A\cos 3x - 9B\sin 3x$

$= 9Cxe^{3x} + 6Ce^{3x} - 9A\cos 3x - 9B\sin 3x$

$\therefore \cancel{9Cxe^{3x}} + 6Ce^{3x} - 9A\cos 3x - 9B\sin 3x - \cancel{9xCe^{3x}} - 9A\cos 3x - 9B\sin$

$= e^{3x} + \sin 3x$

$C = \frac{1}{6}$, $B = -1/18$, $A = 0$

$\therefore y_p = \frac{1}{6} x e^{3x} - \frac{1}{18} \sin 3x$

$y = c_1 e^{-3x} + c_2 e^{3x} + \frac{1}{6} x e^{3x} - \frac{1}{18} \sin 3x$

Ex: Solve $y'' + 4y = x \sin x$, $y(0) = 0$, $y'(0) = 1$

Sol: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$

$y_p = (A_1 x + A_0) \cos x + (B_1 x + B_0) \sin x$

$y_p' = -(A_1 x + A_0) \sin x + A_1 \cos x + (B_1 x + B_0) \cos x + B_1 \sin x$

$y_p'' = -(A_1 x + A_0) \cos x - 2A_1 \sin x - (B_1 x + B_0) \sin x + 2B_1 \cos x$

$\therefore 3(A_1 x + A_0) \cos x + 3(B_1 x + B_0) \sin x - 2A_1 \sin x + 2B_1 \cos x = x \sin$

$\boxed{A_1 = 0}$, $\boxed{B_0 = 0}$, $\boxed{A_0 = -2/9}$, $\boxed{B_1 = 1/3}$

$\therefore y_p = \frac{-2}{9} \cos x + \frac{1}{3} x \sin x$

$y = C_1 \cos 2x + C_2 \sin 2x - \frac{2}{9} \cos x + \frac{1}{3} x \sin x$

at $x = 0$, $y = 0 \Rightarrow 0 = C_1 + 0 - \frac{2}{9} + 0 \Rightarrow \boxed{C_1 = \frac{2}{9}}$

$y' = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{2}{9} \sin x + \frac{1}{3} x \cos x + \frac{1}{3} \sin$

at $x = 0$, $y'(0) = 1 \Rightarrow 1 = 0 + 2C_2 - 0 + 0 + 0 \Rightarrow \boxed{C_2 = \frac{1}{2}}$

$\therefore$ the complete solution is

$y = \frac{2}{9} \cos 2x + \frac{1}{2} \sin 2x - \frac{2}{9} \cos x + \frac{1}{3} x \sin x$

2. Inverse Operator (D- operator) Method

Consider the following 2nd order ODE :-

$$a_2 y'' + a_1 y' + a_0 y = r(x)$$

$$(a_2 D^2 + a_1 D + a_0) y = r(x) \quad , D = \frac{d}{dx} \quad , \frac{1}{D} = \int (\quad) d$$

$$[f(D)] y = r(x)$$

where :-

$$f(D) = a_2 D^2 + a_1 D + a_0$$

the particular Solution can be defined by :

$$\boxed{y_p = \frac{1}{F(D)} r(x)}$$

Rule	$r(x)$	y_p
1	$k e^{a.x}$	$\frac{1}{F(D)\vert_{D=a}} k e^{ax}$ if $F(D)\vert_{D=a} \neq 0$ $x \frac{1}{F'(D)\vert_{D=a}}$ if $F(D)\vert_{D=a} = 0$
2	$k \cos a.x$ $k \sin ax$	$\frac{1}{F(D^2)\vert_{D^2=-a^2}} \begin{Bmatrix} k\cos ax \\ k\sin ax \end{Bmatrix}$ if $F(D^2)\vert_{D^2=-a^2} \neq 0$ $x \cdot \frac{1}{F'(D^2)\vert_{D^2=-a^2}} \begin{Bmatrix} k\cos ax \\ k\sin ax \end{Bmatrix}$ if $F(D^2)\vert_{D^2=-a^2} = 0$
3	x^m	$(1 + a_1 D + a_2 D^2 + a_3 D^3 + \cdots\cdots) x^m$ by Long division

Rule	r(x)	y_p
4.	$e^{ax} u(x)$	$e^{ax} \frac{1}{F(D+a)} u(x)$ e^{ax} is taken out to the left and F(D) is replaced by F(D+a)
5	$x\, u(x)$	$x \frac{1}{F(D)} u(x) - \frac{F'(D)}{[F(D)]^2} u(x)$

Ex:- Solve $y'' - y' - 2y = e^{3x}$ [Rule 1]

Sol:- $y_h = c_1 e^{-x} + c_2 e^{2x}$

$F(D) = D^2 - D - 2$

$$y_p = \frac{1}{D^2-D-2}\Big|_{D=3} e^{3x} = \frac{1}{3^2-3-2} e^{3x} = \frac{1}{4} e^{3x}$$

Ex:- Solve $y'' - 4y' + 4y = 4e^{2x}$ [Rule 1]

Sol:- $y_h = c_1 e^{2x} + c_2 x e^{2x}$

$F(D) = D^2 - 4D + 4$

$$y_p = \frac{1}{D^2-4D+4}\Big|_{D=2} 4e^{2x} = \frac{1}{0} 4e^{2x}$$

$$= x \cdot \frac{1}{2D-4}\Big|_{D=2} 4e^{2x} = \frac{1}{0} 4e^{2x}$$

$$= x^2 \frac{1}{2}\Big|_{D=2} 4e^{2x} = 2x^2 e^{2x}$$

$\therefore\ y = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 e^{2x}$

Ex:- Solve $y'' + 2y' + 3y = \sin x$ [Rule 2]

Sol:- $y_h = c_1 e^{-x} + c_2 e^{-2x}$

$$F(D) = D^2 + 2D + 3$$

$$y_p = \frac{1}{D^2+2D+3}\Bigg|_{D^2=-1} \sin x = \frac{1}{-1+2D+3}\sin x = \frac{1}{2(D+1)}\sin x$$

$$= \frac{1}{2}\,\frac{D-1}{(D+1)(D-1)}\sin x = \frac{1}{2}\,\frac{D-1}{D^2-1}\Bigg|_{D^2=-1}\sin x$$

$$= \frac{1}{2}\cdot\frac{1}{-2}(D-1)\sin x = \frac{-1}{4}(\cos x - \sin x)$$

$$\therefore y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{4}(\cos x - \sin x)$$

Ex:- Find y_p for $y'' + 5y' + 4y = -2x + 3$ [Rule 3]

Sol:- $F(D) = D^2 + 5D + 4 = 4\left(1 + \frac{5}{4}D + \frac{1}{4}D^2\right)$

$$\therefore y_p = \frac{1}{4}\,\frac{1}{1+\frac{5}{4}D+\frac{1}{4}D^2}(-2x+3)$$

$$\begin{array}{r|l} & 1 - \frac{5}{4}D \\ \hline 1+\frac{5}{4}D+\frac{1}{4}D^2 & \cancel{1} \\ & \cancel{1} + \frac{5}{4}D + \frac{1}{4}D^2 \\ \hline & -\cancel{\frac{5}{4}D} - \frac{1}{4}D^2 \\ & -\cancel{\frac{5}{4}D} - \frac{25}{16}D^2 - \frac{5}{16}D^3 \\ \hline \end{array}$$

$$\therefore y_p = \frac{1}{4}\left(1 - \frac{5}{4}D\right)(-2x+3)$$

$$= \frac{1}{4}\left[-2x + 3 + \frac{5}{2}\right]$$

$$= -\frac{x}{2} + \frac{11}{8}$$

H.W: check by undermined Coefficient method

Ex:- Solve $y'' + y' + y = xe^{x} + e^{x}\sin x$ [Rule 4, 5]

Sol:- $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore y_h = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

$F(D) = D^2 + D + 1$

$$y_p = \frac{1}{D^2+D+1}\, xe^{x} + \frac{1}{D^2+D+1}\, e^{x}\sin x$$

$$= x \cdot \frac{1}{D^2+D+1}\Big|_{D=1} e^{x} - \frac{2D+1}{(D^2+D+1)^2}\Big|_{D=1} e^{x} + e^{x}\frac{1}{(D+1)^2+(D+1)+1}\sin x$$

$$= x\,\frac{1}{3}e^{x} - \frac{1}{9}(2D+1)e^{x} + e^{x}\frac{1}{D^2+3D+3}\Big|_{D^2=-1}\sin x$$

$$= \frac{1}{3}xe^{x} - \frac{1}{9}(2e^{x}+e^{x}) + e^{x}\frac{3D-2}{(3D+2)(3D-2)}\sin x$$

$$= \frac{1}{3}xe^{x} - \frac{1}{3}e^{x} + e^{x}\frac{3D-2}{9D^2-4}\Big|_{D^2=-1}\sin x$$

$$= \frac{1}{3}e^{x}(x-1) + e^{x}\cdot\frac{-1}{13}(3\cos x - 2\sin x)$$

$$\therefore y = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{3}e^{x}(x-1) - \frac{1}{13}e^{x}(3\cos x - 2\sin x)$$

Ex: Find y_p for $y'' + 4y = x\sin x$ [Rule 5]

Sol:- $F(D) = D^2 + 4$

$$y_p = \frac{1}{D^2+4}\, x\sin x = x\,\frac{1}{D^2+4}\Big|_{D^2=-1}\sin x - \frac{2D}{[D^2+4]^2}\Big|_{D^2=-1}\sin x$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}(D)\sin x$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}\cos x$$

3. Variation of Parameters Method

Variation of parameter is a general method for find the particular solution y_p of linear ODEs.

Consider the following 2nd order ODE:-

$$a_2 y'' + a_1 y' + a_0 y = r(x)$$

the solution is: $y(x) = y_h(x) + y_p(x)$

Let $y_h = c_1 u_1(x) + c_2 u_2(x)$

$y_p = v_1(x) u_1(x) + v_2(x) u_2(x)$

the particular solution y_p can be obtained by solving the following equations:-

$$v_1' u_1 + v_2' u_2 = 0$$

$$v_1' u_1' + v_2' u_2' = r(x)$$

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

$$\Delta = u_1 u_2' - u_1' u_2$$

$$v_1' = \frac{-u_2\, r(x)}{\Delta} \Rightarrow \boxed{v_1 = -\int \frac{u_2\, r(x)}{\Delta}\, dx}$$

$$v_2' = \frac{u_1\, r(x)}{\Delta} \Rightarrow \boxed{v_2 = \int \frac{u_1\, r(x)}{\Delta}\, dx}$$

EX :- Solve $y'' - 4y' + 3y = \frac{1}{1+e^{-x}}$

Sol :- $\lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 3, \ \lambda_2 = 1 \Rightarrow y_h = c_1 e^{3x} + c_2 e^{x}$

$\therefore u_1 = e^{3x} \Rightarrow u_1' = 3e^{3x}$

$u_2 = e^{x} \Rightarrow u_2' = e^{x}$

$\therefore \Delta = e^{3x} \cdot e^{x} - 3e^{3x} \cdot e^{x} = -2e^{4x}$

$$V_1 = -\int \frac{e^{x} \cdot \frac{1}{1+e^{-x}}}{-2e^{4x}}\, dx = \frac{1}{2}\int \frac{e^{-3x}}{(1+e^{-x})}\, dx$$

let $z = 1 + e^{-x} \Rightarrow dz = -e^{-x}\, dx$

$$\therefore V_1 = \frac{-1}{2}\int \frac{(z-1)^2}{z}\, dz = \frac{-1}{2}\int \left(z - 2 + \frac{1}{z}\right) dz$$

$$= \frac{-1}{2}\left(\frac{z^2}{2} - 2z + \ln z\right)$$

$$V_1 = \frac{-1}{2}\left[\frac{1}{2}(1+e^{-x})^2 - 2(1+e^{-x}) + \ln(1+e^{-x})\right]$$

$$V_2 = \int \frac{e^{3x}\, \frac{1}{1+e^{-x}}}{-2e^{4x}}\, dx = \frac{1}{2}\int \frac{-e^{-x}}{1+e^{-x}}\, dx$$

$$= \frac{1}{2}\ln(1+e^{-x})$$

$\therefore y_p = u_1 V_1 + u_2 V_2$

$$= -\frac{e^{3x}}{2}\left[\frac{1}{2}(1+e^{-x})^2 - 2(1+e^{-x}) + \ln(1+e^{-x})\right]$$

$$+ \frac{1}{2} e^{x} \ln(1+e^{-x})$$

$y = y_h + y_p =$

EX Solve $y'' + y = \tan x$, $y(0) = 1$, $y'(0) = 2$

Sol :- $y_h = c_1 \cos x + c_2 \sin x$

$u_1 = \cos x \Rightarrow u_1' = -\sin x$

$u_2 = \sin x \Rightarrow u_2' = \cos x$

$\Delta = \cos x \cdot \cos x - (-\sin x) \sin x = 1$

$$V_2 = \int \frac{\cos x \tan x}{1} dx = \int \sin x \, dx = -\cos x$$

$$V_1 = -\int \frac{\sin x \cdot \tan x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx = -\int \sec x \, dx - \int \cos x \, dx$$

$$= -\ln|\sec x + \tan x| + \sin x$$

$\therefore y_p = \sin x (-\cos x) + \cos x(-\ln|\sec x + \tan x| + \sin x)$

$= -\sin x \cancel{\cos x} - \cos x \left[-\ln(\sec x + \tan x|\right] + \sin x \cancel{\cos x}$

$\therefore y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

$y = 1$ at $x = 0 \Rightarrow 1 = c_1 + 0 - 0 \Rightarrow \boxed{c_1 = 1}$

$$y' = -c_1 \sin x + c_2 \cos x - \cos x \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$+ \sin x \ln|\sec x + \tan x|$$

$y' = 2$ at $x = 0$

$2 = 0 + c_2 - 1 \Rightarrow \boxed{c_2 = 3}$

$\therefore y = \cos x + 3 \sin x - \cos x \ln|\sec x + \tan x|$

Variation of Parameters

This method assumes we already know the homogeneous solution

$$y_h = C_1 u_1(x) + C_2 u_2(x)$$

The method consists of replacing the constants C_1 and C_2 by functions $v_1(x)$ and $v_2(x)$ and then requiring that the new expression

$$y_h = v_1 u_1 + v_2 u_2$$

and by solving the following two equations

$$v_1' u_1 + v_2' u_2 = 0$$

$$v_1' u_1' + v_2' u_2' = F(x)$$

for the unknown functions v_1' and v_2' using the following matrix notation

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ F(x) \end{bmatrix}$$

Finally v_1 and v_2 can be found by integration.

In applying the method of ***variation of parameters*** to find the particular solution, the following steps are taken:

i. Find v_1' and v_2' using the following equations

$$v_1' = \frac{\begin{vmatrix} 0 & u_2 \\ F(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{-u_2 F(x)}{D}, \qquad v_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & F(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 F(x)}{D}$$

where $$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

ii. Integrate v_1' and v_2' to find v_1 and v_2.

iii. Write the particular solution as

$$y_p = v_1 u_1 + v_2 u_2$$

Example

Solve the equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 6$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

The characteristic equation is $D^2 + 2D - 3 = 0$ and the roots of this equation are $r_1 = -3$ and $r_2 = 1$, so

$$y_h = C_1 e^{-3x} + C_2 e^{x}$$

Then $$u_1 = e^{-3x},\ u_2 = e^{x}$$

$$D = \begin{vmatrix} e^{-3x} & e^{x} \\ -3e^{-3x} & e^{x} \end{vmatrix} = e^{-2x} + 3e^{-2x} = 4e^{-2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{x} \\ 6 & e^{x} \end{vmatrix}}{4e^{-2x}} = \frac{-6e^{x}}{4e^{-2x}} = -\frac{3}{2}e^{3x}, \qquad v_2' = \frac{\begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & 6 \end{vmatrix}}{4e^{-2x}} = \frac{6e^{-3x}}{4e^{-2x}} = \frac{3}{2}e^{-x}$$

$$v_1 = \int -\frac{3}{2}e^{3x}dx = -\frac{1}{2}e^{3x}, \qquad v_2 = \int \frac{3}{2}e^{-x}dx = -\frac{3}{2}e^{-x}$$

$$y_p = v_1 u_1 + v_2 u_2 = \left(-\frac{1}{2}e^{3x}\right)e^{-3x} + \left(-\frac{3}{2}e^{-x}\right)e^{x} = -2$$

$$y = y_h + y_p = C_1 e^{-3x} + C_2 e^{x} - 2$$

Example

Solve the equation $y'' - 2y' + y = e^{x}\ln(x)$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$D^2 - 2D + 1 = 0$$

$$(D-1)^2 = 0$$

The roots are $$r_1 = r_2 = 1$$

The solution is $$y_h = (C_1 x + C_2)e^{x}$$

$$y_h = C_1 x e^{x} + C_2 e^{x}$$

From that we have $u_1(x) = xe^{x}$, and $u_2(x) = e^{x}$.

$$D = \begin{vmatrix} xe^{x} & e^{x} \\ xe^{x} + e^{x} & e^{x} \end{vmatrix} = xe^{2x} - (xe^{2x} + e^{2x}) = -e^{2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{x} \\ e^{x}\ln(x) & e^{x} \end{vmatrix}}{-e^{2x}} = \frac{-\ln(x)e^{2x}}{-e^{2x}} = \ln(x)$$

$$v_2' = \frac{\begin{vmatrix} xe^{x} & 0 \\ xe^{x} + e^{x} & e^{x}\ln(x) \end{vmatrix}}{-e^{2x}} = \frac{x\ln(x)e^{2x}}{-e^{2x}} = -x\ln(x)$$

$$v_1 = \int \ln(x)dx = x\ln(x) - x$$

$$v_2 = -\int x\ln(x)dx$$

$$u = \ln(x) \quad \Rightarrow \quad du = \frac{dx}{x}, \qquad dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

$$v_2 = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x^2}{2} \times \frac{1}{x}dx\right) = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx\right)$$

$$= -\left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) = \frac{x^2}{4} - \frac{x^2}{2}\ln(x)$$

The particular solution is

$$y_p = v_1u_1 + v_2u_2 = (x\ln(x) - x)xe^x + \left(\frac{x^2}{4} - \frac{x^2}{2}\ln(x)\right)e^x$$

$$= x^2e^x\ln(x) - x^2e^x + \frac{x^2}{4}e^x - \frac{x^2}{2}e^x\ln(x)$$

$$= \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

The complete solution is

$$y = y_h + y_p = C_1xe^x + C_2e^x + \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

Exercises

Find the solution of the following Differential Equations

1) $y'' + y = 3x^2$

2) $y'' + 2y' + y = x^2$

3) $y'' + 2y' + 3y = 27x$

4) $y'' + y = -30\sin(4x)$

5) $y'' + y = 6\sin(x)$

6) $y'' + 4y' + 3y = \sin(x) + 2\cos(x)$

7) $y'' + 4y' + 4y = 18\cosh(x)$

8) $y'' - 2y' + 2y = 2e^x\cos(x)$

9) $y^{(4)} - 5y'' + 4y = 10\cos(x)$

10) $y'' + y' - 2y = 3e^x$

11) $y'' + y = x^2 + x$

12) $y'' - y = e^x$

13) $y'' - 2y' + y = e^x$

14) $y'' + y' + y = x^4 + 4x^3 + 12x^2$

15) $y''' + 2y'' - y' - 2y = 1 - 4x^3$

16) $y'' - 2y' + 2y = 2e^x\cos(x)$