كلية : التربية للعلوم الصرفة

القسم او الفرع : الرياضيات

المرحلة: الثانية

أستاذ المادة : د.فراس شاكر فندي

اسم المادة بالغة العربية: النظرية الزمر

اسم المادة باللغة الإنكليزية: **Groups Theorem**

اسم المحاضرة الثانية باللغة العربية: خوا ص الزمرة

اسم المحاضرة الثانية باللغة الإنكليزية : **properties of group**

Example(6). Let $G = R \times R = \{(a,b) : a, b \in R, a \neq 0\}$ and $*$ be defined by $(a,b)*(c,d) = (ac, bc+d)$. Prove that $(G,*)$ is a group

Solution: Let $(a,b), (c,d), (e,f) \in G \Rightarrow a \neq 0, c \neq 0, e \neq 0$

اغلاق

① **closure:** $a \neq 0, c \neq 0 \Rightarrow a.c \neq 0$

$$(a,b)*(c,d) = (ac, bc+d) \in G$$

تجميعية

② **Associative:**

$$L.H.S = (a,b)*[(c,d)*(e,f)]$$
$$= (a,b)*(ce, de+f)$$
$$\boxed{= (ace, bce+de+f)} \quad \text{------} ①$$

$$R.H.S = [(a,b)*(c,d)]*(e,f)$$
$$= (ac, bc+d)*(e,f)$$
$$= (ace, (bc+d).e+f)$$
$$= \boxed{(ace, bce+de+f)} \quad \text{------} ②$$

③ **Identity:** Let $e = (x,y)$

$(a,b)*(x,y) = (a,b)$

$(ax, bx+y) = (a,b)$

$ax = a \Rightarrow x = 1$

$bx + y = b$ نعوض قيمة x

$b(1) + y = b \Rightarrow$

$b + y = b \Rightarrow y = b - b = 0$

$(x,y)*(a,b) = (a,b)$

$(xa, ya+b) = (a,b)$

$xa = a \Rightarrow x = 1$

$ya + b = b$

$ya = b - b$

$ya = 0 \xRightarrow{\div a} y = 0$

Then $e = (x,y) = (1,0) \in G$

④ **Inverse:** Let $(a,b)^{-1} = (x,y)$

$(a,b)*(x,y) = (1,0)$

$(ax, bx+y) = (1,0)$

$ax = 1$ —① $\Rightarrow x = \frac{1}{a}$

$bx + y = 0$ —②

② نعوض قيمة x في المعادلة

$b(\frac{1}{a}) + y = 0 \Rightarrow \frac{b}{a} + y = 0$

$y = -\frac{b}{a}$

$(x,y) = (\frac{1}{a}, -\frac{b}{a})$

$(x,y)*(a,b) = (1,0)$

$(xa, ya+b) = (1,0)$

$xa = 1 \Rightarrow x = \frac{1}{a}$

$ya + b = 0 \Rightarrow ya = -b \Rightarrow y = -\frac{b}{a}$

$\therefore (x,y) = (\frac{1}{a}, -\frac{b}{a})$

$(a,b)^{-1} = (\frac{1}{a}, -\frac{b}{a}) \in G$ $\quad (\frac{1}{a} \neq 0)$

Then $(G,*)$ is group.

Exercises: Determine the systems $(G,*)$ described abelian (ابدالية) (commutative) group.

① $G = \mathbb{Z}$, $a*b = a+b-2$

② $G = R \times R = \{(a,b) : a \neq 0\}$ s.t
$(a,b)*(c,d) = (ac, b+d)$

③ $(G = \{an : n \in \mathbb{Z}\}, +)$

⑤ $G = Q^+$, $a*b = \frac{a.b}{2}$

Theorem: If G is a group with a binary operation $*$, Then.

① $e^{-1} = e$

② $(a^{-1})^{-1} = a \quad , \forall a \in G$

③ $(a*b)^{-1} = b^{-1}*a^{-1} \quad , \forall a,b \in G$

④ $a*b = a*c \Rightarrow b=c$ } cancellation Laws.

⑤ $b*a = c*a \Rightarrow b=c$

proof:-

② $L.H.S = (a^{-1})^{-1} = (a^{-1})^{-1}*e$

$= (a^{-1})^{-1}*(a^{-1}*a)$

$= ((a^{-1})^{-1}*a^{-1})*a$

$= e*a$

$= a$

للتوضيح

ملاحظة

let $b = a^{-1}$

$b*b^{-1} = e$

$(a^{-1})*(a^{-1})^{-1} = e$

= =

③ $(a*b)^{-1} = b^{-1}*a^{-1} \quad , \forall a,b \in G$

proof:- $a*b \in G \Rightarrow (a*b)^{-1} \in G$

$(a*b)*(a*b)^{-1} = (a*b)^{-1}*(a*b) = e$

$(a*b)*(a*b)^{-1} = e$

$a^{-1}*(a*b)*(a*b)^{-1} = a^{-1}*e$

$(a^{-1}*a)*b*(a*b)^{-1} = a^{-1}$

$e*b*(a*b)^{-1} = a^{-1}$

$b*(a*b)^{-1} = a^{-1}$

$(b^{-1}*b)*(a*b)^{-1} = b^{-1}*a^{-1}$

$e*(a*b)^{-1} = b^{-1}*a^{-1} \Rightarrow \boxed{(a*b)^{-1} = b^{-1}*a^{-1}}$

(4) $a*b=a*c$

$a^{-1}*(a*b)=a^{-1}*(a*c)$

$(a^{-1}*a)*b=(a^{-1}*a)*c \qquad (*\ \text{ass.})$

$e*b=e*c$

$b=c$

= =

(5) $b*a=c*a$

$(b*a)*a^{-1}=(c*a)*a^{-1}$

$b*(a*a^{-1})=c*(a*a^{-1})$

$b*e=c*e$

$b=c$

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Theorem: Let $(G,*)$ be a group, there is only one element e in G such that $e*a=a*e=a$, $\forall a\in G$

proof:- let e & e' are two identity element in G.

$a*e=a \qquad \& \qquad e*a=a$

$a*e'=a \qquad \& \qquad e'*a=a$

$\cancel{a}*e=\cancel{a}*e' \Rightarrow e=e'$

or $e*\cancel{a}=e'*\cancel{a} \Rightarrow e=e'$

= =

Theorem: In a group $(G,*)$, the inverse element of each element in G is unique. وحيد

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proof:- Let $a \in G$ and $a^{-1} = x, x' \in G$

$\therefore \; a*x = e \quad \& \quad x*a = e$

$a*x' = e \quad \& \quad x'*a = e$

$\therefore \; \cancel{a}*x = \cancel{a}*x' \Rightarrow x = x'$

or $x*\cancel{a} = x'*\cancel{a} \Longrightarrow x = x'$

Theorem: Let $(G,*)$ be a group, the equations المعادلات $\boxed{a*x=b}$ and $\boxed{y*a=b}$ have لها a unique وحيد solution. حل

proof:-

$a*x = b \Rightarrow$

$a^{-1}*(a*x) = a^{-1}*b$

$(a^{-1}*a)*x = (a^{-1}*b)$

$e*x = a^{-1}*b$

$x = (a^{-1}*b)$

$y*a = b$

$(y*a)*a^{-1} = b*a^{-1}$

$y*(a*a^{-1}) = (b*a^{-1})$

$y*e = b*a^{-1}$

$y = (b*a^{-1})$

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② Let $(G,*)$ is a group such that $a^2 = e, \forall a \in G$ show that $(G,*)$ is a commutative group.