الجامعة	الانبار
الكلية	العلوم
القسم	الرياضيات
المرحلة	الرابعة
اسم المادة باللغة العربية	التحليل العقدي
اسم المادة باللغة الانكليزية	Complex Analysis
اسم التدريسي	دكتورة دنيا علاوي جروان
عنوان المحاضرة باللغة العربية	المتسلسلات العقدية
عنوان المحاضرة باللغة الإنكليزية	Complex Series
رقم المحاضرة	L9

Taylor's series for complex analytic functions.

Taylor's theorem:-

If $f(z)$ is analytic function inside the circle $C: |z-z_0| = r$, then for each z inside C,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{, where}$$

$$a_n = \frac{f^{n}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-z_0)^{n+1}}$$

when $z_0 = 0$, then the series is called Maclaurin Series.

Remark (1): we can write $f(z)$ in taylor's series as:

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + f'''(z_0)\frac{(z-z_0)^3}{3!} + \dots + f^{(n)}(z_0)\frac{(z-z_0)^n}{n!} + \dots$$

Remark (2) Taylor's series is convergent series its summation $f(z)$, when $|z-z_0| < r$.

Example: write taylor series for the function:-

$f(z) = \frac{1}{1-z}$ about $z = 0$

Sol// $f(z)$ is analytic function on $\mathbb{C}/\{1\}$, we can choose $C: |z| = 1$, then $\frac{1}{1-z}$ is analytic inside C, that is in the region $|z| < 1$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} a_n (z-0)^n = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n = \frac{f^{n}(0)}{n!}$, $\forall n = 0, 1, 2, \dots$ (1)

$$f(0) = 1 \quad , \quad f'(z) = \frac{1}{(1-z)^2} \Rightarrow f'(0) = 1$$

$$f''(z) = \frac{2(1-z)}{(1-z)^4} = \frac{2}{(1-z)^3} \Rightarrow f''(0) = 2 = 2!$$

$$f'''(z) = \frac{6(1-z)^2}{(1-z)^6} = \frac{6}{(1-z)^4} \Rightarrow f'''(0) = 6 = 3!$$

$$f''''(z) = \frac{24(1-z)^3}{(1-z)^8} = \frac{24}{(1-z)^5} \Rightarrow f''''(0) = 24 = 4!$$

$$\therefore f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow f^{n}(0) = n!$$

$$\therefore \frac{1}{1-z} = f(0) + f^{(1)}(0)z + \frac{f^{(2)}(0)z^2}{2!} + \frac{f^{(3)}(0)z^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} z^n$$

$$= 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

1/ Find The Taylor series for $\frac{1}{1+z}$ at $z_0=0$

Sol $f(z) = \frac{1}{1+z} \Rightarrow f(0) = \frac{1}{1+0} = 1$

$f^{(1)}(z) = \frac{-1}{(1+z)^2} \Rightarrow f^{(1)}(0) = -1 = -(1!)$ odd

$f^{(2)}(z) = \frac{2\cancel{(1+z)}}{(1+z)^{\cancel{4}}} = \frac{2}{(1+z)^3} \Rightarrow f^{(2)}(0) = \frac{2}{1} = 2!$ even

$f^{(3)}(z) = \frac{-6\cancel{(1+z)^2}}{(1+z)^{\cancel{6}}} = \frac{-6}{(1+z)^4} \Rightarrow f^{(3)}(0) = -6 = -(3!)$ odd

$f^{(4)}(z) = \frac{24\cancel{(1+z)^3}}{\cancel{(1+z)^8}} = \frac{24}{(1+z)^5} \Rightarrow f^{(4)}(0) = 24 = 4!$ even

so $f^{(n)}(z) = \frac{(-1)^n n!}{(1+z)^{n+1}} \Rightarrow f^{(n)}(0) = (-1)^n n!$

$\therefore \frac{1}{1+z} = f(0) + \frac{f^{(1)}(0)\, z}{1!} + \frac{f^{(2)}(0)\, z^2}{2!} + \frac{f^{(3)}(0)\, z^3}{3!} + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n z^n$

$= 1 - z + z^2 - z^3 + z^4$

$= \cancel{1} + \frac{(-1)\cancel{1!}}{\cancel{1!}} z + \frac{\cancel{2!}}{\cancel{2!}} z^2 + \frac{(-1)\cancel{3!}}{\cancel{3!}} z^3$

$+ \frac{\cancel{4!}}{\cancel{4!}} z^4 + \cdots$

Ex// Find the Taylor series for $f(z)=e^{z}$ at $z_0=0$

Sol:- $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$= \sum_{n=0}^{\infty} a_n (z-0)^n = \sum_{n=0}^{\infty} a_n z^n$

$f(0)=e^{0}=1$

$f'(z)=e^{z} \Rightarrow f'(0)=1$

$f^{(2)}(z)=e^{z} \Rightarrow f^{(2)}(0)=1$

$f^{(3)}(z)=e^{z} \Rightarrow f^{(3)}(0)=1$

$\vdots$

$f^{(n)}(z)=e^{z} \Rightarrow f^{(n)}(0)=1$

$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!}$

Hence
$$f(z)=e^{z}=\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Example:- Find the Maclaurin Series for $f(z)=\sin z$

Sol
$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$
$$= \sum_{n=0}^{\infty} a_n z^n \quad ; \quad z_0=0$$

$f(0)=\sin 0 = 0$

$f^{(1)}(z)=\cos z \Rightarrow f^{(1)}(0)=\cos(0)=1$

$f^{(2)}(z)=-\sin z \Rightarrow f^{(2)}(0)=-\sin(0)=0$

$$f^{(3)}(z) = -\cos z \Rightarrow f^{(3)}(0) = -\cos(0) = -1$$

$$f^{(4)}(z) = \sin z \Rightarrow f^{(4)}(0) = \sin(0) = 0$$

$$f^{(5)}(z) = \cos z \Rightarrow f^{(5)}(0) = \cos(0) = 1$$

if n $\begin{cases} \to \text{even } 2n \Rightarrow f^{2n}(0) = 0 \Rightarrow a_n = \dfrac{f^{2n}(0)}{(2n)!} = 0 \\ \to \text{odd } 2n+1 \Rightarrow f^{2n+1}(0) = (-1)^n \Rightarrow \end{cases}$

$$a_n = \frac{f^{2n+1}(0)}{(2n+1)!} = \frac{(-1)^n}{(2n+1)!}$$

$$\therefore f(z) = \sin z = \sum_{n=0}^{\infty} \frac{f^{2n}(0)}{(2n)!}(z-0)^{2n} + \sum_{n=0}^{\infty} \frac{f^{2n+1}(0)}{(2n+1)!}(z-0)^{2n+1}$$

$$f(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

Example: Find the Taylor Series for

$$f(z) = \frac{1}{(1-z)(z+4)}$$

Sol since $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ and

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$\Rightarrow \frac{1}{(1-z)(z+4)} = \frac{A}{1-z} + \frac{B}{z+4} \Rightarrow \frac{A(z+4)+B(1-z)}{(1-z)(z+4)}$$

$$\Rightarrow \frac{Az - Bz + 4A + B}{(1-z)(z+4)} = \frac{(A-B)z + 4A + B}{(1-z)(z+4)}$$

$$\Rightarrow A - B = 0 \Rightarrow A = B$$

$$4A + B = 1 \Rightarrow 4A + A = 1 \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$$

So

$$\frac{1}{(1-z)(z+4)} = \frac{1}{5}\left(\frac{1}{1-z}+\frac{1}{z+4}\right) = \frac{1}{5}\left(\frac{1}{1-z}+\frac{1}{4(1+\frac{z}{4})}\right)$$

$$= \frac{1}{5}\left(\sum_{n=0}^{\infty} z^n + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n\right)$$

$$= \frac{1}{5}\left(\sum_{n=0}^{\infty} z^n + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{4^n}\right)$$

$$= \frac{1}{5}\left(\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{4^{n+1}}\right)$$

$$= \frac{1}{5}\left(\sum_{n=0}^{\infty}\left(1+\frac{(-1)^n}{4^{n+1}}\right) z^n\right)$$

Ex/ Find the Taylor series of $f(z) = \cos z$ at $z_0 = \pi$

Sol

$f(z) = \cos z \Rightarrow f(\pi) = \cos(\pi) = -1$

$f'(z) = -\sin z \rightarrow f^{(1)}(\pi) = -\sin(\pi) = 0$

$f^{(2)}(z) = -\cos z \Rightarrow f^{(2)}(z) = -\cos(\pi) = 1$

$f^{(3)}(z) = \sin z \Rightarrow f^{(3)}(\pi) = \sin(\pi) = 0$

$f^{(4)}(z) = \cos z \Rightarrow f^{(4)}(\pi) = \cos(\pi) = -1$

$f^{(5)}(z) = -\sin z \Rightarrow f^{(5)}(\pi) = -\sin(\pi) = 0$

If n $\begin{cases} \text{odd } 2n+1 \Rightarrow f^{(2n+1)}(\pi) = 0 \Rightarrow a_n = \dfrac{f^{(2n+1)}(\pi)}{(2n+1)!} = 0 \\ \text{even } 2n \rightarrow f^{(2n)}(\pi) = (-1)^n \Rightarrow a_n = \dfrac{f^{(2n)}(\pi)}{(2n)!} = \dfrac{(-1)^n}{(2n)!} \end{cases}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$(z) = \cos z = \sum_{n=0}^{\infty} \frac{f^{2n}(\pi)}{(2n)!}(z-\pi)^{2n} + \sum_{n=0}^{\infty} \frac{f^{2n+1}(\pi)}{(2n+1)!}(z-\pi)^{2n+1}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-\pi)^{2n}$$

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عنوان المحاضرة باللغة العربية	متسلسلة لورنت
عنوان المحاضرة باللغة الإنكليزية	Laurent Series
رقم المحاضرة	L10

Laurent's series مسلسلة لورانت

سوف ندرس امكانية إيجاد مفكوك دالة ذات نمط أعم من سلاسل القوى وسلسلة تايلور والتي تحتوي على قوى صحيحة موجبة وسالبة لـ $(z-z_0)$ تسمى هذه السلسلة لورانت وصيغتها

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad , n=0,\mp 1,\mp 2,\ldots$$

such that $\quad a_n = \frac{1}{2\pi i}\int_c \frac{f(s)\,ds}{(s-z_0)^{n+1}}$

$$c: \; 0<|z-z_0|<r_1$$

ويمكن كتابتها بالصيغ

$$f(z)=\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principle part}}$$

and $\quad b_n = \frac{1}{2\pi i}\int f(s)(s-z_0)^{n-1}\,ds$

ملاحظة: ١- اذا كان لدالة ما نقطة منفردة معزولة z_0 فان تلك الدالة تتمثل بسلسلة لورانت .

Ex// The series of the form $\sum_{n=0}^{\infty} z^{n-1}$ is Laurent series.

since $\quad \sum_{n=0}^{\infty} z^{n-1} = \underbrace{z^{-1}}_{\text{principle part}} + \underbrace{z^{0} + z + z^{2} + z^{3} + \cdots}_{\text{analytic part}}$

Example: Find Laurent series for the function $\frac{e^z - 1}{z}$ at $z_0 = 0$

Sol

$$f(z) = \frac{e^z - 1}{z}$$

since $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$

So

$$f(z) = \frac{\left(\cancel{1} + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - \cancel{1}}{z}$$

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Example :- Find the Laurent series for

$$f(z) = \frac{z - \cos z}{z} \quad \text{at } z_0 = 0$$

Sol : $\cos z = \sum_{n=0}^{\infty} (-1)^{2n} z^{2n}/(2n)!$

$$f(z) = \frac{z - \left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]}{z}$$

$$= \frac{z - 1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots}{z}$$

$$= 1 - \frac{1}{z} + \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} + \cdots$$

So $$\frac{z - \cos z}{z} = 1 - \frac{1}{z} + \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} + \cdots$$

Example Find Laurent series for $f(z) = \frac{1}{(z+1)(z+2)}$

sol 1-

$$\frac{1}{(z+1)(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+2)}$$

$$\Rightarrow A = 1 \quad , \quad B = 1$$

Hence

$$\frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

when $|z| > 1 \longrightarrow \left|\frac{1}{z}\right| < 1$

$$\frac{1}{z+1} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$$

when $|z| \leq 2$

$$\Rightarrow$$

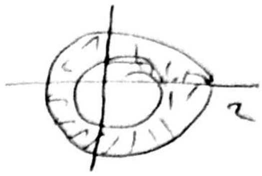
$$\frac{1}{z+2} = \frac{1}{2(1+\frac{z}{2})} = \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right)$$

$$\frac{1}{z+2} = \frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \cdots$$

$$\therefore \frac{1}{z+1} - \frac{1}{z+2} = \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4}\right) - \left(\frac{1}{2} - \frac{z}{4} + \frac{z^2}{8} - \frac{z^3}{16} + \cdots\right)$$

$$= \cdots - \frac{1}{z^4} + \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z} - \frac{1}{2} + \frac{z}{4} - \frac{z^2}{8} + \frac{z^3}{16} + \cdots$$

Example 1- Expland the function $f(x) = \frac{-1}{(z-1)(z-2)}$ in the form of Laurent's series in the ring $1<|z|<2$



Sol

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

since

$$\frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z+(-2A-B)}{(z-1)(z-2)}$$

$\therefore A+B=0 \Rightarrow A=-B$ and
$$-2A-B=-1$$
$$-2(-B)-B=-1$$
$$2B-B=-1$$
$$B=-1$$

So
$$\frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$\forall\ |z|>1$

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = \frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$
$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$
$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$\forall\ |z|<2$

$$-\frac{1}{z-2} = \frac{1}{2-z} = \frac{1}{2(1-\frac{z}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
$$= \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\left(\frac{z}{2}\right)^n\right]$$
$$= \frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\cdots = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$\therefore f(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

Ex/. If: $f(z) = \dfrac{\ln(z)}{(z-1)^5}$, find Laurent series for $f(z)$.

Sol $z-1=0 \Rightarrow z=1$ singular point

$\ln(z)$ has Taylor series at $z=1$

$$\ln(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} \quad \text{at } z=0$$

at $z=1$

$$\ln z = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1}$$

$$\therefore f(z) = \frac{\ln(z)}{(z-1)^5} = \frac{\sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1}}{(z-1)^5}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1-5}}{(n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n-4}}{n+1}$$