الجامعة	الانبار
الكلية	العلوم
القسم	الرياضيات
المرحلة	الرابعة
اسم المادة باللغة العربية	التحليل العقدي
اسم المادة باللغة الانكليزية	Complex Analysis
اسم التدريسي	دكتورة دنيا علاوي جروان
عنوان المحاضرة باللغة العربية	الدوال التوافقية
عنوان المحاضرة باللغة الإنكليزية	Hormonic Functions
رقم المحاضرة	L7

Hormonic Functions — الدوال التوافقية.

Definition: Let $f(x,y)$ be a real function in two variables x,y. Then f is said to be hormonic function in a region D, if the partial derivative of the first and second order of f w.r.t x and y are continuous in D and satisfy Laplace's equation.

$$f_{xx} + f_{yy} = 0.$$

Example: Let $f(x,y) = 2xy$. To prove that f is hormonic

$f_x = 2y \quad , f_y = 2x$

$f_{xx} = 0 \quad , f_{yy} = 0.$

Since f_x, f_y, f_{xx}, f_{yy} are continuous

Also

$$f_{xx} + f_{yy} = 0 + 0 = 0$$

So $f(x,y)$ is hormonic function

Remark:- Let $f(z) = u + iv$ be analytic function. Then the partial derivative of u and v of all orders are exist and continuous, also from real analysis we obtain that:

$u_{xy} = u_{yx} \wedge v_{xy} = v_{yx}$ because the continuouity of the higher derivative.

Theorem 6:- If $f(z) = u + iv$ is an analytic function on a region D, then each of $u(x,y)$ and $v(x,y)$ is a hormonic function.

Proof: $f(z)$ is analytic function on D, then f is differentiable on D.

Hence u and v satisfy C.R.E in D.
so $u_x = v_y$ and $v_x = -u_y$

$\Rightarrow u_{xx} = v_{yx}$ and $v_{xy} = -u_{yy}$

$\Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{xy}$

$= v_{xy} - v_{xy} = 0$ since $v_{yx} = v_{xy}$
from above Remark

So $u_{xx} + u_{yy} = 0$
and by the same way, we get $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$

Therefore u and v are hormonic function

Definition: Let each of $u(x,y)$ and $v(x,y)$ be hormonic in a region D. If u and v satisfy C.R.E, then we say that v is hormonic conjugate for u (يسمى v المرافق التوافقي لـ u)

The following Theorem gives a necessary and sufficient condition for a function to be analytic
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Theorem 7: A function $f(z) = u(x,y) + iv(x,y)$ is an analytic function in D iff v is hormonic conjugate of u in D.

Proof: suppose that $f(z)$ is analytic function
T.P v is hormonic conjugate of u

Then by Theorem 6 and Definition of a hormonic conjugate, we get V is a hormonic conjugate for U.

$\Leftarrow$ If V is a hormonic conjugate of U, then by Definition of hormonic conjugate each of U, V is a hormonic function and the partial derivatives of U and V w.r.t x and y are continuous and satisfy Cauchy-Riemann equations on D.

Hence $f(z)$ is differentiable on D.

So $f(z)$ is analytic function on D.

Remark: ① If $f(z)$ is analytic function in D, then V is a hormonic conjugate for U.

② If V is a hormonic conju for U, then $f(z)$ is analytic function in D.

③ If V is a hormonic conjugate for U in region D, then it's not necessary that U is a hormonic conjugate for V,

Example:- $f(z) = z^2$, since $f(z)$ is analytic function in $\mathbb{C}$ $\Rightarrow$ V is a hormonic conj. for U. But U is not a hormonic conj. for V. $f(z) = x^2 - y^2 + 2xy$

$u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$\Rightarrow v_x = u_y \Rightarrow 2y = -2y \Rightarrow y = 0$

$v_y = -u_x \Rightarrow 2x = -2x \Rightarrow x = 0$

المساواة تحقق فقط عندما $x=y=0$ أي في النقطة $(0,0)$.

So u is not a hormonic conjugate for v.

Example:- Find the hormonic conjugate for the function $u(x,y)=y^3-3x^2y$

solution:- $u_x=-6xy$, $u_{xx}=-6y$

$u_y=3y^2-3x^2 \Rightarrow u_{yy}=6y$

$u_{xx}+u_{yy}=-6y+6y=0$

$\therefore u$ is hormonic.

$u_x=-6xy$

$\because u_x=v_y$, $u_y=-v_x$

$v_y=-6xy$

$v=\int -6xy\,dy+\phi(x)$

$=-3xy^2+\phi(x)$

$v_x=-3y^2+\phi'(x)=-u_y=-(3y^2-3x^2)$

$=-3y^2+3x^2$

So $\cancel{-3y^2}+\phi'(x)=\cancel{-3y^2}+3x^2$

$\phi'(x)=3x^2$

$\phi(x)=\int 3x^2\,dx=x^3+c$

$v(x,y)=-3xy^2+x^3+c$

when $c=0$

$v(x,y)=-3xy^2+x^3$.

$f(z)=y^3-3x^2y+i(x^3-3xy^2)$