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اسم المادة باللغة العربية	التحليل العقدي
اسم المادة باللغة الانكليزية	Complex Analysis
اسم التدريسي	دكتورة دنيا علاوي جروان
عنوان المحاضرة باللغة العربية	معادلتي كوشي ريمان في الصيغة القطبية
عنوان المحاضرة باللغة الإنكليزية	Polar Form of Cauchy Riemann Equations
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Polar form of Cauchy Riemann Equations

Let $Z = r(\cos\theta + i\sin\theta)$, where $x = r\cos\theta$, $y = r\sin\theta$
and $f(Z) = u(r,\theta) + iv(r,\theta)$.
derive C.R.E in polar form.

الصيغه القطبية لمعادلتي كوشي

$u_r = \frac{1}{r} v_\theta$, $v_r = \frac{-1}{r} u_\theta$

يمكن اشتقاق هذه الصيغه كما يلي

$$\frac{df}{dZ} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial Z}$$

$$\because Z = r e^{i\theta} \Rightarrow \frac{\partial Z}{\partial r} = e^{i\theta} \Rightarrow \frac{\partial r}{\partial Z} = e^{-i\theta}$$

Since $f = u + iv \Rightarrow \frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$

so $\frac{df}{dZ} = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$ ---- (1)

and also

$$\frac{dF}{dZ} = \frac{\partial F}{\partial \theta} \cdot \frac{\partial \theta}{\partial Z}$$

$$\because Z = r e^{i\theta} \Rightarrow \frac{\partial Z}{\partial \theta} = i r e^{i\theta} \Rightarrow \frac{\partial \theta}{\partial Z} = \frac{1}{ir} e^{-i\theta} = -\frac{i}{r} e^{-i\theta}$$

also $F = u + iv \Rightarrow \frac{\partial F}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

Hence

$$\frac{dF}{dZ} = -\frac{i}{r} e^{-i\theta}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] \quad \text{.....} \quad (2)$$

بمساواة (1) مع (2) نحصل على

$$\cancel{e^{-i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = -\frac{i}{r}\cancel{e^{-i\theta}}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right]$$

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

i.e $\Rightarrow U_r + iV_r = -\frac{i}{r}U_\theta + \frac{1}{r}V_\theta$

$\Rightarrow U_r = \frac{1}{r}V_\theta \quad , \quad V_r = \frac{-1}{r}U_\theta$

and

$$f'(z) = e^{-i\theta}\left[U_r + iV_r\right] = e^{-i\theta}\left[U_r(r,\theta) + iV_r(r,\theta)\right]$$

$$= e^{-i\theta}\left[\frac{1}{r}V_\theta - i\frac{1}{r}U_\theta\right]$$

$$= \frac{1}{r}e^{-i\theta}\left[V_\theta - iU_\theta\right].$$

Example: Let $f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$

$\Rightarrow \frac{1}{r}(\cos\theta - i\sin\theta)$

$\Rightarrow U(r,\theta) = \frac{\cos\theta}{r} \quad , \quad V(r,\theta) = \frac{-\sin\theta}{r}$

$\Rightarrow U_r = \frac{-\cos\theta}{r^2} \quad , \quad U_\theta = \frac{-\sin\theta}{r}$

$\Rightarrow V_r = \frac{\sin\theta}{r^2} \quad , \quad V_\theta = \frac{-\cos\theta}{r}$

C.R.E are hold since

$U_r = \frac{1}{r}V_\theta$ and $V_r = \frac{-1}{r}U_\theta$

and all partial derivatives U_x, U_y, V_x and V_y are continuous So f' exists and

$$f'(z) = e^{-i\theta}\left[U_r + iV_r\right] = e^{-i\theta}\left[\frac{-\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right]$$

$$= \frac{e^{-i\theta}}{r^2}\left(-e^{-i\theta}\right) = \frac{-1}{r^2e^{2i\theta}} = \frac{1}{z^2}$$

Example:- Consider the function $g(z) = \sqrt{r}\, e^{i\theta/2}$, $r > 0$, $0 < \theta < \pi$. Show that $g(z)$ has derivative at each point in its domain of definition, and $g'(z) = \frac{1}{2g(z)}$

$$g(z) = \sqrt{r}\, e^{i\theta/2} = r^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= r^{1/2}\cos\frac{\theta}{2} + i r^{1/2}\sin\theta/2$$

so $u = r^{1/2}\cos\frac{\theta}{2}$, $v = r^{1/2}\sin\theta/2$

$\Rightarrow$ $u_r = \frac{1}{2} r^{-1/2}\cos\frac{\theta}{2}$, $u_\theta = \frac{-1}{2} r^{1/2}\sin\frac{\theta}{2}$

$v_r = \frac{1}{2} r^{-1/2}\sin\frac{\theta}{2}$, $v_\theta = \frac{1}{2} r^{1/2}\cos\frac{\theta}{2}$

C.R.E are hold and continuous

$\therefore$ $g'(z)$ exists.

$$g'(z) = e^{-i\theta}[u_r + i v_r]$$

$$= e^{-i\theta}\left[\frac{1}{2} r^{-1/2}\cos\frac{\theta}{2} + i\frac{1}{2} r^{-1/2}\sin\frac{\theta}{2}\right]$$

$$= e^{-i\theta}\cdot\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= e^{-i\theta}\cdot\frac{1}{2\sqrt{r}} e^{i\theta/2} = \frac{1}{2\sqrt{r}} e^{-i\theta/2}$$

$$= \frac{1}{2\sqrt{r}\, e^{i\theta/2}}$$

$$= \frac{1}{2g(z)}.$$

Theorem 5:- If $f(z)$ is differentiable in region D in a complex plane $\mathbb{C}$, and $f'(z)=0$, $\forall z \in D$, then f is constant function in D.

Theorem 6:- If f is analytic in region D in a complex plane $\mathbb{C}$, and $f'(z)=0$, $\forall z \in D$, the f is a constant function in D.

Example: If f is analytic function in the domain D. and $\operatorname{Im} f$ is constant, then f is constant.

proof:- let $f(z)=u+iv$ is analytic in D

$\because \operatorname{Im} f$ is constant $\Rightarrow v$ is constant

$$\therefore \frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$$

i.e $v_x=v_y=0$

$\because f$ is analytic function in D

$\therefore f$ is defferentiable in D

$\Rightarrow$ Cauchy-Riemann Equation are satisfied

$\therefore u_x=v_y$ and $u_y=-v_x$

$\because v_x=v_y=0 \Rightarrow u_x=u_y=0$

$\therefore u$ is constant by Theorem 5

$\therefore f=u+iv$ is constant function