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اسم المادة باللغة العربية	التحليل العقدي
اسم المادة باللغة الانكليزية	Complex Analysis
اسم التدريسي	دكتورة دنيا علاوي جروان
عنوان المحاضرة باللغة العربية	الدوال التحليلية
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Analytic Function الدوال التحليلية

Definition: A function f is said to be analytic at a point z_0 if f is differentiable at z_0 and there exists a neighbourhood of z_0 such that f is differentiable at each point of this neighbourhood.

Definition:- A function f is said to be analytic in a region D if f is analytic at each point of D.

Definition: A point z_0, is called a singular point for a function f if f is not analytic at z_0, but f is analytic at some points of each neighbourhood of z_0.

Definition:- A singular point for a function f is called isolated singular point if there exists a neighbourhood of z_0 such that f is analytic at each of its points except at the point z_0 itself.

Example:- Let $f(z) = \dfrac{z-3}{z(z^2+4)}$

$z(z^2+4)=0 \Rightarrow$ ~~Either~~ $z=0$ or $z^2+4=0$

$\Rightarrow z^2=-4 \Rightarrow z=\mp 2i$

So $z=0, \mp 2i$ are isolated singular points

That is f is analytic at $\mathbb{C} \setminus \{0, \mp 2i\}$.

Remark:- There exist a singular point, but is not isolated point.

6, 7, 8, 9, ~~10~~

Definition:- A function f is said to be entire function (دالة كلية) if it is analytic at each point of the complex plane.

Example:- $f(z) = 3z - \frac{1}{3}z + 1$ is analytic at each point of $\mathbb{C}$, so f is entire function

Example: $f(z) = e^{x}$ is entire function

Example: $f(z) = |z|^2$ is not analytic at $z=0$.

Cauchy - Rieman Equations

معادلتي كوشي - ريمان

Theorem 3:- (Cauchy - Rieman).

If the function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z_0 = x_0 + iy_0$, then the partial derivatives u_x, u_y, v_x and v_y must exist at (x_0, y_0) and they must satisfy the Cauchy Rieman equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Also
$$f(z_0) = u_x + iv_x$$
$$= v_y - iu_y$$

proof:-

since $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z_0 = x_0 + iy_0$

$$\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

writing $\Delta Z = \Delta x + i\Delta y$, we have.

$$f(Z_0+\Delta Z) = f\big((x_0+iy_0)+(\Delta x+i\Delta y)\big)$$
$$= f\big(x_0+\Delta x + i(y_0+\Delta y)\big)$$
$$= u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y)$$

and $f(Z_0) = u(x_0,y_0) + iv(x_0,y_0)$

So $f(Z_0+\Delta Z) - f(Z_0) = \left[u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)\right]$

$$+ i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right]$$

Hence

$$f'(Z_0) = \lim_{\Delta Z\to 0} \frac{f(Z_0+\Delta Z) - f(Z_0)}{\Delta Z}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x+\Delta y} + i \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)}{\Delta x+\Delta y} \quad \text{----}(1)$$

Now
Let $\Delta Z \to 0$ through the real axis
so $\Delta y \to 0$ and $\Delta Z = \Delta x$
The equation (1) became

$$f'(z_0) = \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

since $u_x = \dfrac{\partial u}{\partial x} = \lim_{\Delta x\to 0} \dfrac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}$

and

$v_x = \dfrac{\partial v}{\partial x} = \lim_{\Delta x\to 0} \dfrac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$

So $f'(z_0) = u_x + i v_x$ ---- (2)

similarly let $\Delta Z \to 0$ through the immaginary axis.

So $\Delta x = 0$ and $\Delta z = i\Delta y$

Therefore (1) become

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$= -i u_y + v_y$$

$$f'(z_0) = v_y - i u_y \quad \text{-- (3)}$$

from (2) and (3)

$$f'(z_0) = u_x + i v_x$$
$$= v_y - i u_y$$

we have the cauchy-Rieman equations

$u_x = v_y$ and $u_y = -v_x$.

Example:- The function $f(z) = z^2$ is differentiable at every point of $\mathbb{C}$.

Since $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xy\,i$

$\Rightarrow u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$u_x = 2x$ and $v_x = 2y$

$u_y = -2y$ and $v_y = 2x$

So $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ Cauchy Rieman equations are satisfied.

Remark:- The converse of (3) is not true in general. By the following example:

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ (\bar{z})^2 / z & \text{if } z \neq 0 \end{cases}$$

Since Cauchy-Rieman Equation hold at $z=0$, however f is not diff. at $z=0$.

الان نعطي الشرط الكافي (sufficient condition) لكي تكون الدالة قابلة للاشتقاق من خلال نظرية (4) حيث أن الدالة التي تحقق معادلتي كوشي ريمان ليست بالضرورة أن تكون قابلة للاشتقاق.

Theorem 4: If $f(z) = u(x,y) + i v(x,y)$ is defined at $z_0 = x_0 + i y_0$ in some neighbourhood of z_0 such that the first order partial derivatives of u and v with respect to x and y are defined in the neighbourhood and continuous at (x_0, y_0) and satisfy Cauchy Rieman equations at (x_0, y_0), then f is differentiable at z_0.

Example:- $f(z) = e^x \cos y + i e^x \sin y$. Is $f'(z)$ exist.

Solution:

$u(x,y) = e^x \cos y \longrightarrow u_x = e^x \cos y \,,\; u_y = -e^x \sin y$

and $v(x,y) = e^x \sin y \longrightarrow v_x = e^x \sin y \,,\; v_y = e^x \cos y$

Note that: $u_x = v_y$ and $v_x = -u_y$ and u_x, u_y, v_x and v_y are continuous at whole plane $\mathbb{C}$, so $f'(x)$ exists at $\mathbb{C}$ and $f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$

Example:- $f(z) = \bar{z}$. Is $f(z)$ diff

Solution:

$f(z) = x - iy$; $z = x + iy$

$u(x,y) = x \rightarrow u_x = 1$, $u_y = 0$

$v(x,y) = -y \rightarrow v_x = 0$ $v_y = -1$

$u_x \neq v_y$

$\therefore f(z)$ is not differentiable on plane.

$\longleftrightarrow$