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Derivatives المشتقات

Definition: Let $f(z)$ be a complex-valued function defined in a neighbourhood of z_0. Then the derivatives of $f(z)$ at z_0 is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

where $\Delta z = z - z_0 \Rightarrow \Delta z = \Delta x + i\Delta y$
and $\Delta x = (x - x_0)$, $\Delta y = (y - y_0)$.

Remark. The function f is said to be differentiable at z_0 if $f'(z_0)$ exists.

Example: prove that $f(z) = 3z^2$ at the point $(0,0)$.

Solution:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z+\Delta z)^2 - 3z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(z^2 + 2z\Delta z + (\Delta z)^2) - 3z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{3z^2} + 6z\Delta z + 3(\Delta z)^2 \cancel{- 3z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(6z + 3\Delta z)}{\cancel{\Delta z}} = \lim_{\Delta z \to 0} 6z + 3\Delta z$$

$$f'(z) = 6z$$

$$f'(0) = 0$$

and f is differentiable in all $\mathbb{C}$.

Example:- Prove that the following function

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \bar{z}^2/z & \text{if } z \neq 0 \end{cases}$$, is not differentiable.

solution:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$f'(0) = \lim_{z \to 0} \frac{(\bar{z}^2/z) - 0}{z} = \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^2$$

if $z \to 0$ on the x-axis, then $y = 0$

$$f'(0) = \lim_{x \to 0} \left(\frac{x}{x}\right)^2 = 1$$

if $z \to 0$ through the line $x = y$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x - iy}{x + iy}\right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y - iy}{y + iy}\right)^2 = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{\cancel{y}(1-i)}{\cancel{y}(1+i)}\right)^2$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(1-i)^2}{(1+i)^2} = \frac{(1-i)^2}{(1+i)^2} = \frac{1 - 2i - 1}{1 + 2i - 1} = -1$$

Therefore $f'(0)$ is not exist

Example:-

orem 1 If f is differentiable function at apoint z_0, then f is continuous at z_0, but he converse is not true in general.

proof: Since f is adifferentiable function at z_0, therefore $f'(z_0)$ exists and

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z \to z_0}$$

we have to show that $\lim_{z \to z_0} f(z) = f(z_0)$

$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{(z - z_0)} \cdot (z - z_0) \right)$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0)$$

$$= f'(z) \cdot 0 = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) \cdot f(z_0)$$

Hence f is a continuous function at z_0.

The converse is not true ingeneral. Consider the following Example:-

$f(z) = |z|^2$ is continuous function at each point of $\mathbb{C}$, but is not differentiable only at $z = 0$

To prove $f(z) = |z|^2$ is continuous function

$$f(z) = |z|^2 = x^2 + y^2$$

$$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$$

each of u and v a cont. functions on $\mathbb{C}$

$\therefore f(z)$ is cont. function on $\mathbb{C}$.

To show that $f(z) = |z|^2$ is not differentiable only at $z = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z\bar{z}} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\overline{\Delta z}}{\Delta z} + \frac{(\bar{z}+\overline{\Delta z})\cancel{\Delta z}}{\cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}$$

If $z = 0 \Rightarrow f'(z) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$ exists at $z = 0$

To show that f is not differentiable at any p[oint] only at $z = 0$

If $\Delta z \to 0$ through the x-axis then $\Delta z = \overline{\Delta z}$, $\Delta y = 0$

So $f'(z) = \lim_{\Delta z \to 0} z\cancel{\frac{\overline{\Delta z}}{\Delta z}} + \bar{z} + \overline{\Delta z} = z + \bar{z} = L_1$

If $\Delta z \to 0$ through y-axis $\Rightarrow \Delta x = 0$

$\Rightarrow \Delta z = -\overline{\Delta z}$

and $f'(z) = \lim_{\Delta z \to 0} \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}\right)$

$$= \lim_{\Delta z \to 0} z\frac{-\Delta z}{\Delta z} + \bar{z} + \overline{\Delta z} = -z + \bar{z} = L_2$$

$L_1 \neq L_2$ so $f'(z)$ doesn't exist

Theorem 2:

1. If k is a continuous, then $\frac{d}{dz}(k) = 0$
2. If $f(z) = z^n$, then $f'(z) = n z^{n-1}$
3. $\frac{d}{dz}(k f(z)) = k f'(z)$, where f is differentiable.
4. If f and g are differentiable functions then
   (i) $\frac{d}{dz}(f(z)+g(z)) = f'(z) + g'(z)$
   (ii) $\frac{d}{dz}(f(z)\, g(z)) = f(z)\, g'(z) + f'(z)\, g(z)$.
   (iii) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z) f'(z) - f(z) g'(z)}{(g(z))^2}$; $g(z) \neq 0$
5. If $F(z) = g(f(z))$ composit function then:
$$F'(z) = g'(f(z))\, f'(z)$$
under the Condition g is differentiable at $f(z)$.