الجامعة	الانبار
الكلية	العلوم
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المرحلة	الرابعة
اسم المادة باللغة العربية	التحليل العقدي
اسم المادة باللغة الانكليزية	Complex Analysis
اسم التدريسي	دكتورة دنيا علاوي جروان
عنوان المحاضرة باللغة العربية	الاستمرارية
عنوان المحاضرة باللغة الإنكليزية	Continuity
رقم المحاضرة	L٣

Continutily الاستمرارية

Definition: Let $f(z)$ be a function defined in some neighbourhood of the point z_0. Then f is said to be continuous function at z_0 iff $\lim_{z \to z_0} f(z) = f(z_0)$, that is $f(z)$ is

continuous at z_0 iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, when $|z - z_0| < \delta$

Definition: A function f is continuous at z_0 iff

(1) $f(z_0)$ exists (2) $\lim_{z \to z_0} f(z)$ exists

(3) $\lim_{z \to z_0} f(z) = f(z_0)$

Definition: A function f is said to be continuous in D if it is continuous at each point of D.

Example: $f(z) = z$ is cont.

Example: $f(z) = \frac{1}{z}$ is discontinuous at $z = 0$

Example: $f(z) = z^2$ is continuous at $z_0 = 3$ since

$f(3)$ is exists since $f(3) = 3^2 = 9$

$\lim_{z \to 3} z^2 = 3^2 = 9$ exists and

$\lim_{z \to 3} z^2 = f(3) \Rightarrow f(z)$ is cont. at $z_0 = 3$

~~Example~~: prove that $f(z) = z + 1$ is cont. at $z = 1$ by def.

$\therefore |z+1-2| < \epsilon$ when $|z-1| < \delta$

$|z-1| < \epsilon$ when $|z-1| < \delta$

$\therefore \epsilon = \delta$

$\therefore f$ is continuous.

Example: let $f(z) = \frac{z+i}{z-i}$. is f cont. at $z = 3i$

solution. $z - i = 0 \Rightarrow z = i$

$D: \mathbb{C} \setminus \{i\}$

$$\lim_{z \to 3i} \frac{z+i}{z-i} = \frac{3i+i}{3i-i} = \frac{4i}{2i} = 2$$

$$f(3i) = \frac{3i+i}{3i-i} = \frac{4i}{2i} = 2.$$

and $\lim_{z \to 3i} f(z) = f(3i)$ $\therefore f$ is cont at $z = 3i$.

Theorem 4:- A function $f(z) = u + iv$ is continuous at the point $z_0 = x_0 + iy_0$ iff the functions u and v are continuous at the point (x_0, y_0).

Proof:- suppose that $f(z)$ is continuous at z_0 then $\lim_{z \to z_0} f(z) = f(z_0)$ by def of cont.

$$\lim_{z \to z_0} (u + iv) = f(z_0) = u_0 + iv_0$$

$\Rightarrow \lim_{(x,y) \to (x_0,y_0)} u = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v = v_0$ by (th. 2)

$\therefore$ each of u and v are continuous at $(x_0, y_0) = z_0$

$\Leftarrow$ suppose that each of u and v are cont. at the point (x_0, y_0)

$\therefore \lim\limits_{(x,y)\to(x_0,y_0)} u = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v = v_0$ by def of cont.

Now

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} (u + iv)$$

$$= \lim_{(x,y)\to(x_0,y_0)} u + i \lim_{(x,y)\to(x_0,y_0)} v$$

$$= u_0 + iv_0$$

$$\therefore \lim_{z\to z_0} f(z) = f(z_0)$$

$\therefore f(z)$ is cont. at z_0

Theorem 5: If $f(z)$ is a continuous function at z_0, then $|f(z)|$ is also continuous

proof:- Since $f(z)$ is cont at z_0, then

$\lim\limits_{z\to z_0} f(z) = f(z_0)$ by def of cont.

and since $\lim\limits_{z\to z_0} |f(z)| = |\lim\limits_{z\to z_0} f(z)| = |f(z_0)|$

$\therefore |f(z)|$ is cont at z_0.

Remark: If $f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ be a polynom in z, then $f(z)$ is continuous in $\mathbb{C}$.

Theorem 6: If $f(Z)$ is continuous on a closed and bounded region D, then $f(Z)$ is a bounded function on D.

proof:- since $f(Z)$ is continuous function on D

$\therefore |f(Z)|$ is continuous on D (by Theorem 5).

Let $f(Z) = u + iv$, then $|f(Z)| = \sqrt{u^2+v^2}$ is continuous, that is the real function $\sqrt{u^2+v^2}$ is continuous on a closed and bounded region D.

$\therefore |f(Z)|$ has a maximum value in D.

That is there exist a positive real number M such that $|f(Z)| \leq M \;\forall Z \in D$.

$\therefore f(Z)$ is a bounded function on D. $\square$

Theorem 7: Let $f(Z)$ and $g(Z)$ be continuous functions at Z_0. Then:

1- $f(Z) \mp g(Z)$ is cont. at Z_0.
2- $f(Z) \cdot g(Z)$ is cont. at Z_0
3- $\frac{f(Z)}{g(Z)}$ is cont. at Z_0, such that $g(Z) \neq 0$.

Theorem 8: let $f(Z)$ be a function defined on a neighbourhood B of the point Z_0 and let $g(Z)$ be a function defined on a region D, such that $f(B) \subset D$. If f cont. at Z_0 and g cont. at $f(Z_0)$, then the composite function $g \circ f$ is continuous at Z_0

$f(Z_0)$

Example: Show that the function $f(z) = xy^2 + i2xy$ is cont. at every where.

Solution :-

$$u(x,y) = xy^2 \quad \text{and} \quad v(x,y) = 2xy$$

Since the two real functions u and v are polynomall with respect to x and y, and since the polynomail is cont. every where. Hence u is cont. function and v is cont. function

By Theorem 4. we get f(z) is cont. ∎

Example prove that the function $f(z) = |z|$ is cont on its domian. by def of cont.

proof :

Let $\epsilon > 0$, to find $\delta > 0$ such that if $|z - z_0| < \delta$, $\forall z$, then $|f(z) - f(z_0)| < \epsilon$

$$\left| f(z) - f(z_0) \right| = \big| |z| - |z_0| \big| < \epsilon$$
$$= |z - z_0| < \epsilon$$
$$\Rightarrow \delta = \epsilon$$