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اسم المادة باللغة العربية	التحليل العقدي
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Limits النهايات

Def:- $\lim_{z \to z_0} f(z) = w_0$ mean $\forall \epsilon > 0 \ \exists$ $\delta > 0$ such that $|f(z) - w_0| < \epsilon$, whenever $0 < |z - z_0| < \delta$

Example: prove that $\lim_{z \to 2i-1} (2z+3) = 4i+1$

Solution: $\forall \epsilon > 0$, we must find $\delta > 0$ such that $0 < |z - (2i-1)| < \delta \Rightarrow |2z+3 - (4i+1)| < \epsilon$

so

$$|2z+3-4i-1| < \epsilon$$

$$|2z-4i+2| < \epsilon$$

$$|2(z-(2i-1))| < \epsilon$$

$$2|z-(2i-1)| < \epsilon$$

$$|z-(2i-1)| < \frac{\epsilon}{2}$$

so $\delta = \epsilon/2$

Example: prove that $\lim_{z \to 1} \frac{iz}{2} = i/2$ by definition

Solution:- $\forall \epsilon > 0, \exists \delta > 0$ s.t $|z - z_0| < \delta$ $\Rightarrow |f(z) - w_0| < \epsilon$

$$\left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon \quad ; |z-1| < \delta$$

$$\left|\frac{i}{2}(z-1)\right| < \epsilon$$

$$\left|\frac{i}{2}\right| |z-1| < \epsilon$$

$$\frac{1}{2}|z-1| < \epsilon \quad \text{since } |i| = 1$$

$$|z-1| < 2\epsilon$$

so $\delta = 2\epsilon$

Example:- prove that $\lim\limits_{z\to 1} \frac{iz}{2} = \frac{i}{2}$ by definition.

Sol// $\forall \epsilon > 0, \exists \delta > 0, |z - z_0| < \delta \Rightarrow$

$|f(z) - w_0| < \epsilon \Rightarrow \left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon, |z-1| < \delta$

$\left|\frac{i}{2}(z-1)\right| < \epsilon \Rightarrow \left|\frac{i(z-1)}{2}\right| < \epsilon$

since $|i| = 1$

$\Rightarrow \left|\frac{z-1}{2}\right| < \epsilon$

$\Rightarrow |z-1| < 2\epsilon$

So $\delta = 2\epsilon$

Example: If $f(z) = \frac{iz^2}{2}$ prove that

$\lim\limits_{z\to 1} f(z) = \frac{i}{2}$, by definition

Sol// $\forall \epsilon > 0, \exists \delta > 0$, such that $|z - z_0| < \delta$

$|f(z) - w_0| < \epsilon$ when $|z-1| < \delta$

$\Rightarrow \left|\frac{iz^2}{2} - \frac{i}{2}\right| < \epsilon \Rightarrow \left|\frac{i(z^2-1)}{2}\right| < \epsilon$

$\left|\frac{z^2-1}{2}\right| < \epsilon \Rightarrow \frac{|z^2-1|}{2} < \epsilon$

$\frac{|(z-1)(z+1)|}{2} < \epsilon \Rightarrow \frac{|z-1|\,|z+1|}{2} < \epsilon$

so $z \to 1$ So $\frac{|z-1|\,|1+1|}{2} < \epsilon$

$\Rightarrow \frac{|z-1|\cancel{2}}{\cancel{2}} < \epsilon \Rightarrow |z-1| < \epsilon$

إذن $\delta = \epsilon$

Example: prove that $\lim_{z\to 2i} z^2 = -4$ by definition.

Sol// $\forall \epsilon > 0, \exists \delta > 0$ such that

if $|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$

so $|z^2 + 4| < \epsilon$; $|z - 2i| < \delta$

$|z^2 - 4i^2| < \epsilon \Rightarrow |(z-2i)(z+2i)| < \epsilon$

$\because z \to 2i$

so $|z - 2i||2i + 2i| < \epsilon$

$|z - 2i||4i| < \epsilon \Rightarrow |z - 2i||4| < \epsilon$

$4|z - 2i| < \epsilon \Rightarrow |z - 2i| < \frac{\epsilon}{4}$

so $\delta = \epsilon/4$

Example:- show that $\lim_{z\to 0} f(z)$ does not exist where $f(z) = \frac{\bar{z}}{z}$

Solution: $\lim_{z\to z_0} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$, $z_0 = (0,0)$

اذا كان الاقتراب خلال المحور الحقيقي $y = 0$

$\Rightarrow \lim_{x\to 0} \frac{x}{x} = 1 = L_1$

اذا كان الاقتراب خلال المحور العمودي اي ان $y\to 0$ $(x=0)$

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} \frac{-iy}{iy} = 1 = L_2$

so $L_1 \neq L_2 \Rightarrow \lim f(z)$ does not exist.

Theorem 1: when a limit of a function f exists at a point z_0, it is unique.

Proof:- suppose that $\lim_{z \to z_0} f(z) = w_0$

and $\lim_{z \to z_0} f(z) = w_1$

T.P $w_0 = w_1$

By def. of limit:

Let $\epsilon > 0$, since $\lim_{z \to z_0} f(z) = w_0$

$\therefore \exists\ \delta_1 > 0$ such that if $|z - z_0| < \delta_1$ then $|f(z) - w_0| < \frac{\epsilon}{2}$

also since $\lim_{z \to z_0} f(z) = w_1$

$\therefore \exists\ \delta_2 > 0$ such that if $|z - z_0| < \delta_2$ then

$$|f(z) - w_1| < \frac{\epsilon}{2}$$

Let $\delta = \min\{\delta_1, \delta_2\}$ T.P $w_0 = w_1$

$$|w_1 - w_0| = |(w_0 - f(z)) + (f(z) - w_1)|$$
$$\leq |w_0 - f(z)| + |f(z) - w_1| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so $|w_1 - w_0| < \epsilon$

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so $w_0 = w_1$

$\therefore$ limit is unique

Theorem 2: in p.9

Theorem 3: Let $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = w_1$

then

(i) $\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z)$

$= w_0 + w_1$

(ii) $\lim_{z\to z_0}[f(z)\cdot g(z)] = \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} g(z) = w_0 \cdot w_1$

(iii) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \dfrac{w_0}{w_1}$, $w_1 \neq 0$

(iv) If $P(z) = a_0 + a_1 z + a_2 z + \dots + a_n z^n$
when $a_0, a_1, \dots, a_n$ complex constants, then
$\lim_{z\to z_0} P(z) = P(z_0)$

proof (i)
suppose $\lim_{z\to z_0} f(z) = w_0$, $\lim_{z\to z_0} g(z) = w_1$

Let $\epsilon > 0$, since $\lim_{z\to z_0} f(z) = w_0$

$\therefore \exists\ \delta_1 > 0$, such that if $|z - z_0| < \delta_1$
then $|f(z) - w_0| < \epsilon/2$

Also when $\lim_{z\to z_0} g(z) = w_1$
$\Rightarrow \exists\ \delta_2 > 0$ such that if $|z - z_1| < \delta$
then $|g(z) - w_1| < \epsilon/2$
Let $\delta = \min\{\delta_1, \delta_2\}$ $\therefore |z - z_0| < \delta$

$|w_0 - w_1| =$

$= \left| f(z) + g(z) - (w_0 + w_1) \right| = \left| f(z) - w_0 + g(z) - w_1 \right|$

$\leq |f(z) - w_0| + |g(z) - w_1|$

$\leq \frac{\epsilon}{2} + \epsilon/2 = \epsilon$

proof (iii)

لكن

$f(z) = f_1 + i f_2$ and $g(z) = g_1 + i g_2$
$w_0 = a_1 + i a_2$ and $w_1 = b_1 + i b_2$

$$\Rightarrow \frac{f(z)}{g(z)} = (f_1 + i f_2) \cdot \left(\frac{g_1}{g_1^2 + g_2^2} - i \frac{g_2}{g_1^2 + g_2^2} \right)$$

$$= \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} + i \frac{(g_1 f_2 - f_1 g_2)}{g_1^2 + g_2^2}$$

وفي جوار قريب حول z_0 تكون $g(z) \neq 0$ أي

$g_1^2 + g_2^2 \neq 0$

وحالة $f_1 \to a_1$, $f_2 \to a_2$

also $g_1 \to b_1$, $g_2 \to b_2$ (by theo 2)

also $b_1^2 + b_2^2 \neq 0$

Since $w_1 \neq 0$ when $z \to z_0$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{a_1 b_1 + a_2 b_2}{b_1^2 + b_2^2} + i \frac{b_1 a_2 - a_1 b_2}{b_1^2 + b_2^2}$$

$$= \frac{a_1 + i a_2}{b_1 + i b_2} = \frac{w_0}{w_1}$$

Theorem 2: Let $f(z) = u(x,y) + iv(x,y)$, then $\lim_{z \to z_0} f(z) = u_0 + iv_0 = w_0$ iff $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$

and $\lim_{z \to z_0} v(x,y) = v_0$ when $z_0 = x_0 + iy_0$

$, w_0 = u_0 + iv_0$

Proof:- If $\lim_{z \to z_0} f(z) = u_0 + iv_0$ T.P $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0$

and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$

By definition of limit $\forall \epsilon > 0, \exists \delta > 0$, such that $|f(z) - w_0| < \epsilon$, when $|z - z_0| < \delta$

Hence

$$|f(z) - w_0| = |u + iv - (u_0 + iv_0)| = |(u - u_0) + i(v - v_0)| < \epsilon$$

$\forall z$ which satisfied.

$$0 < |z - z_0| = |x + iy - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| < \delta$$

and since

$$|u - u_0| \leq |f(z) - w_0| < \epsilon$$

$$|v - v_0| < |f(z) - w_0| < \epsilon$$

$\forall x, y$ which satisfied

$$|x - x_0| < |\cancel{x_0} + x) + i(y - y_0)| < \delta$$

$$|y - y_0| < |(x - x_0) + i(y - y_0)| < \delta$$

$\therefore \lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$

$\Leftarrow$ H.W

Theorem (2) Let $f(z) = u(x,y) + iv(x,y)$, then

$$\lim_{z\to z_0} f(z) = u_0 + iv_0 = w_0 \quad \text{iff} \quad \lim_{\substack{x\to x_0 \\ y\to y_0}} u(x,y) = u_0$$

and $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} v(x,y) = v_0$, when $z_0 = x_0 + iy_0$

proof: