UNIVERSITY OF ANBAR

كلية : التربية للعلوم الصرفة

القسم او الفرع : الرياضيات

المرحلة: الاولى

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اسم المادة بالغة العربية : التفاضل والتكامل

اسم المادة باللغة الإنكليزية : **Calculus**

اسم المحاضرة السادسة باللغة العربية: (المشتقات)

اسم المحاضرة السادسة باللغة الإنكليزية : **(Derivative)**

Chapter three

Derivatives

8- Derivative using the definition

The Derivative of the function $y = f(x)$ is the function $y' = f'(x)$ whose value at each x is define by rule

$$y = f(x) \Rightarrow y' = f'(x)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Definition: If $y = f(x)$ is a continuous function, then we define the derivative of function as a limit as

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example:- Find the derivative of the function $y = x^2$ by define.

Solve:

$y = f(x) = x^2$ **and**

$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$ **, then**

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[2x + \Delta x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

9- The Rules for Derivative

1- If $y = a \implies \frac{dy}{dx} = 0$, where a is constant.

Example: $y = 2 \implies \frac{dy}{dx} = 0$.

2- If $y = x^n \implies \frac{dy}{dx} = nx^{n-1}$, where n any number.

Example: $y = x^{-2} \implies \frac{dy}{dx} = -2x^{-2-1} = -2x^{-3}$.

3- If $y = ax^n \implies \frac{dy}{dx} = a \cdot nx^{n-1}$.

Example: $y = 4\sqrt[3]{x} \implies \frac{dy}{dx} = 4 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = \frac{4}{3\sqrt[3]{x^2}}$.

4- If $y = u(x) + v(x) \implies \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Example: $y = 2x^2 + 8 - 5x^4 \implies \frac{dy}{dx} = 4x + 0 - 20x^3 = 4x - 20x^3$.

5- If $y = b[u(x)]^n \implies \frac{dy}{dx} = b \cdot n[u(x)]^{n-1} \cdot \frac{du}{dx}$ where b is constant.

Example: $y = 3(2x^2 - x + 4)^7 \implies \frac{dy}{dx} = 3 \cdot 7(x^2 - x + 4)^6 \cdot (4x - 1)$

6- If $y = u(x) \cdot v(x) \Rightarrow \frac{dy}{dx} = u(x) \cdot \frac{dv}{dx} + v(x) \cdot \frac{du}{dx}$

Example: $y = (x^2+1)(x-3)^2 \Rightarrow (x^2+1)[2(x-3)] + (x-3)^2[2x]$

7- If $y = \frac{u(x)}{v(x)} \Rightarrow \frac{dy}{dx} = \frac{v(x)\cdot\frac{du}{dx} - u(x)\cdot\frac{dv}{dx}}{[v(x)]^2}$.

Example: $y = \frac{x^2+1}{3x^2+2x} \Rightarrow \frac{dy}{dx} = \frac{(3x^2+2x)\cdot(2x)-(x^2+1)\cdot(6x+2)}{[3x^2+2x]^2}$

$$= \frac{(6x^3+4x^2)-(6x^3+2x^2+6x+2)}{9x^4+12x^3+4x^2} = \frac{2x^2-6x-2}{9x^4+12x^3+4x^2}$$

10- The Derivative of trigonometric functions

1- $y = \sin(g(x)) \Rightarrow y' = \cos(g(x)) \cdot g'(x)$

2- $y = \cos(g(x)) \Rightarrow y' = -\sin(g(x)) \cdot g'(x)$

3- $y = \tan(g(x)) \Rightarrow y' = \sec^2(g(x)) \cdot g'(x)$

4- $y = \cot(g(x)) \Rightarrow y' = -\csc^2(g(x)) \cdot g'(x)$

5- $y = \sec(g(x)) \Rightarrow y' = \sec(g(x))\tan(g(x)) \cdot g'(x)$

6- $y = \csc(g(x)) \Rightarrow y' = -\csc(g(x))\cot(g(x)) \cdot g'(x)$

Example:- Using the definition of the derivative of a function to find the derivative of the functions

1- $f'(x) = x^3 + 2x$

Solve:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - (x^3+2x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 + 2$$

$$= 3x^2 + 2$$

2- $y = \sqrt{x}$

Solve:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{(x + \Delta x)} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{(x + \Delta x)} - \sqrt{x}\right)\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\left(\sqrt{(x + \Delta x)} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Example:- Find the derivatives of the following functions.

1- $f(x) = x^7 - x^{-5} + x^3 - 19 \quad \Rightarrow \quad f'(x) = 7x^6 + 5x^{-6} + 3x^2$

2- $g(x) = x\sqrt{x^2 - 1} \quad \Rightarrow \quad g'(x) = \frac{x \cdot 2x}{2\sqrt{x^2-1}} + \sqrt{x^2 - 1} = \frac{2x^2-1}{\sqrt{x^2-1}}$

3- $y = x + \frac{1}{x^2} \quad \Rightarrow \quad y' = 1 + \frac{-2}{x^3} = 1 - \frac{2}{x^3}$

11- The Derivative of Natural Logarithm functions

If y is function given by $y = \ln(g(x))$, where $g(x) > 0$, then the derivative of y is

$$y = \ln(g(x)) \quad \Rightarrow \quad y' = \frac{g'(x)}{g(x)}$$

For Example:-

$1 - y = \ln(x) \quad \Rightarrow \quad y' = \frac{1}{x}$

$2 - y = \ln(x^2 + 2x) \quad \Rightarrow \quad y' = \frac{2x + 2}{x^2 + 2x}$

12- The Derivative of Exponential functions

The function $e^{g(x)}$ has the derivative given by

$$y = e^{g(x)} \quad \Rightarrow \quad y' = e^{g(x)} \cdot g'(x)$$

For Example:-

$1 - y = e^{x^2-x} \quad \Rightarrow \quad y' = e^{x^2-x} \cdot (2x - 1)$

Example:- Find the derivatives of the following functions.

1- $f(x) = \sin x^2 + \cot(x^4 - 1)$

Solve:

$$f'(x) = 2x\cos x^2 - 4x^3\csc^2(x^4 - 1)$$

2- $g(x) = \sqrt{\csc(x^2) - 1}$

Solve:

$$g'(x) = \frac{-2x\csc x^2 \cot x^2}{2\sqrt{\csc(x^2) - 1}}$$

$$= \frac{-x\csc x^2 \cot x^2}{\sqrt{\csc(x^2) - 1}}$$

3- $y = \ln(2x - x^{-2})$

Solve:

$$y' = \frac{2 + 2x^{-3}}{2x - x^{-2}}$$

4- $f(x) = e^{\frac{1}{x}}$

Solve:

$$\frac{df}{dx} = e^{\frac{1}{x}} \cdot \frac{-1}{x^2}$$

5- $y = \cos(e^{2x})$

Solve:

$$\frac{dy}{dx} = -2e^{2x}\sin(e^{2x})$$

6- $y = (\sec(2x) + \tan(3x))^{-2}$ **H.W.**

7- $g(x) = \ln\sqrt{\frac{1+x}{1-x}}$ **H.W.**

8- $h(x) = x\ln(e^{\cot x})$ **H.W.**

13- The Derivative of $y = a^{g(x)}$, where $a > 0$

If $f(x)$ is a function given in the form $y = f(x) = a^{g(x)}$, then the easiest way to find the derivative y' is by taking logarithms.

$$\ln y = \ln a^{g(x)}$$

$$\Rightarrow \ln y = g(x)\ln a \quad \text{where} \quad \ln a^{g(x)} = g(x)\ln a$$

$$\Rightarrow \frac{y'}{y} = g'(x)\ln a$$

$$\Rightarrow y' = y \cdot g'(x)\ln a$$

$$\Rightarrow y' = a^{g(x)} \cdot g'(x)\ln a \text{ , where } y = a^{g(x)}$$

Thus, if $y = a^{g(x)} \Rightarrow y' = a^{g(x)} \cdot g'(x)\ln a$.

Example:- Find the derivatives of the following functions.

$$1 - f(x) = 2^{(x^4 - x)} \quad \Rightarrow \quad f'(x) = 2^{(x^4 - x)}\,(4x^3 - 1)\ln 2$$

$$2 - y = 7^{(\sin x^2)} \quad \Rightarrow \quad y' = 7^{(\sin x^2)}\,2x\cos x^2 \ln 7$$

$$3 - g(x) = \left(\frac{3}{2}\right)^{\sqrt{x-1}} \quad \Rightarrow \quad g'(x) = \left(\frac{3}{2}\right)^{\sqrt{x-1}} \frac{1}{2\sqrt{x-1}} \ln\left(\frac{3}{2}\right)$$

14- The Derivative of trigonometric reverse functions

$$1 - \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$2 - \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$3 - \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$4 - \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$5 - \frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$6 - \frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

Example:- Find $\frac{dy}{dx}$

1- If $y = \sin^{-1}(2x^2) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{4x}{\sqrt{1-(2x^2)^2}}$

2- If $y = \tan^{-1}(x^2 + 2x)$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2x+2}{1+(x^2+2x)^2}$

3- If $y = \sin^{-1}(x^2 + 3x - \cos(x))$ $\Rightarrow$ $\frac{dy}{dx} = \frac{2x+3-(-\sin(x))}{\sqrt{1-(x^2+3x-\cos(x))^2}}$

4- If $y = \cos^{-1}(x^2 + \tan(2x))$ **H.W.**

Example:- If $y = \sin^{-1}(x)$ **, Prove that** $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Solve:

$$y = \sin^{-1}(x) \Rightarrow x = \sin(y)$$

$$\Rightarrow 1 = \cos(y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Example:- If $y = \cos^{-1}(x)$ **, Prove that** $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ **H.W.**