كلية : التربية للعلوم الصرفة

القسم او الفرع : الرياضيات

المرحلة: الاولى

أستاذ المادة : (م.د. مصطفى ابراهيم حميد) (م.م. عبدالسلام فائق طلك)

اسم المادة بالغة العربية : التفاضل والتكامل

اسم المادة باللغة الإنكليزية : **Calculus**

اسم المحاضرة الرابعة باللغة العربية: (نظرية الغاية)

اسم المحاضرة الرابعة باللغة الإنكليزية : **(Theorem of Limit)**

Chapter two

Theorem of Limit

3- Not the following Rules hold if

$$\lim_{x\to a} f(x) = L \quad and \quad \lim_{x\to a} g(x) = M$$

8- $\lim_{x\to a} c = c \quad , \quad where \quad c \in \mathbb{R}.$

9- $\lim_{x\to a} f(x)c = c\lim_{x\to a} f(x) = cL \quad , \quad where \quad c \in \mathbb{R}.$

10- $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M\,.$

11- $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M\,.$

12- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \quad where\ M \neq 0.$

13- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n \quad , \quad where \quad n \in \mathbb{N}.$

Example:- Evaluate the following Limits.

$$1 - \lim_{x\to 0}\left[\frac{x^4 - x + 1}{x - 1}\right]^3$$

Solve:

$$\lim_{x\to 0}\left[\frac{x^4 - x + 1}{x - 1}\right]^3 = \left[\lim_{x\to 0}\frac{x^4 - x + 1}{x - 1}\right]^3$$

$$= \left[\frac{\lim_{x\to 0} x^4 - \lim_{x\to 0} x + \lim_{x\to 0} 1}{\lim_{x\to 0} x - \lim_{x\to 0} 1}\right]^3 = \left[\frac{0 - 0 + 1}{0 - 1}\right]^3 = -1.$$

$$2-\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]$$

Solve:

$$\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]=\lim_{x\to 5}\left[\frac{x^2-25}{x+5}\right]\lim_{x\to 5}\left[\frac{x^2-25}{x-5}\right]$$

$$=\left[\frac{25-25}{5+5}\right]\lim_{x\to 5}\left[\frac{(x-5)(x+5)}{x-5}\right]$$

$$=\frac{25-25}{5+5}(5+5)=0.$$

$$3-\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}$$

Solve:

$$\lim_{y\to 2}\frac{\sqrt{y^2+12}-4}{y-2}=\lim_{y\to 2}\frac{\left(\sqrt{y^2+12}-4\right)\left(\sqrt{y^2+12}+4\right)}{(y-2)\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{y^2+12-16}{(y-2)\left(\sqrt{y^2+12}+4\right)}=\lim_{y\to 2}\frac{y^2-4}{(y-2)\left(\sqrt{y^2+12}+4\right)}$$

$$=\lim_{y\to 2}\frac{(y-2)(y+2)}{(y-2)\left(\sqrt{y^2+12}+4\right)}=\lim_{y\to 2}\frac{y+2}{\left(\sqrt{y^2+12}+4\right)}$$

$$\frac{4}{4+4}=\frac{1}{2}$$

$$4-\lim_{t\to 4}\frac{t-4}{t^2-t-12} \qquad H.W.$$

$$5-\lim_{x\to -1}\frac{x^3+x+2}{x+1} \qquad H.W.$$

Example:- If $f(x)=x^2-x$ then find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Solve:

Since $f(x)=x^2-x$, $\quad f(x+h)=(x+h)^2-(x+h) \quad and$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-(x+h)-(x^2-x)}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2hx+h^2-x-h-x^2+x}{h}=\lim_{h\to 0}\frac{2hx+h^2-h}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h-1)}{h}=\lim_{h\to 0}(2x+h-1)$$

$$2x+0-1=2x-1$$

4- Infinite Limits

Some times we need to know what happens to $f(x)$ as x gets large and positive $(x\to\infty)$ or large and negative $(x\to-\infty)$ consider a function $f(x)=\frac{1}{x}$ what dose $\lim_{x\to\infty} f(x)$,

$f(x)$ gets close to 0, as x gets large and large , this is written

$$\lim_{x\to\infty}\frac{1}{x}=0 \qquad or \qquad \lim_{x\to-\infty}\frac{1}{x}=0$$

Example:- Find the following Limits if they exist

$$1-\lim_{x\to\infty}\frac{x^3+2x+1}{3x^3+1}$$

Solve:

$$\lim_{x\to\infty}\frac{x^3+2x+1}{3x^3+1}=\lim_{x\to\infty}\frac{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{1}{x^3}}{\frac{3x^3}{x^3}+\frac{1}{x^3}}$$

$$=\lim_{x\to\infty}\frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{3+\frac{1}{x^3}}=\frac{1}{3}$$

$$2-\lim_{x\to\infty}\frac{4x-2}{x^2+3}$$

Solve:

$$\lim_{x\to\infty}\frac{4x-2}{x^2+3}=\lim_{x\to\infty}\frac{\frac{4x}{x^2}-\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{4}{x}-\frac{2}{x^2}}{1+\frac{3}{x^2}}=\frac{0}{1}=0$$

$$3-\lim_{x\to\infty}(x-\sqrt{x^2+x})$$

Solve:

$$\lim_{x\to\infty}(x-\sqrt{x^2+x})=\lim_{x\to\infty}(x-\sqrt{x^2+x})\frac{(x+\sqrt{x^2+x})}{(x+\sqrt{x^2+x})}$$

$$= \lim_{x\to\infty} \frac{(x^2 - (x^2 + x))}{(x + \sqrt{x^2 + x})} = \lim_{x\to\infty} \frac{-x}{(x + \sqrt{x^2 + x})}$$

$$= \lim_{x\to\infty} \frac{-x}{(x + \sqrt{x^2 + x})} = \lim_{x\to\infty} \frac{-\frac{x}{x}}{(\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{x}{x^2}})}$$

$$= \lim_{x\to\infty} \frac{-1}{(1 + \sqrt{1 + \frac{1}{x}})} = \frac{-1}{(1 + \sqrt{1 + 0})} = \frac{-1}{2}$$

$4 - \lim_{x\to\infty} \sqrt{\frac{9x - 1}{x + 1}}$ $\quad H.W.$

$5 - \lim_{x\to\infty} \frac{\sqrt{x^2 + x}}{x + 1}$ $\quad H.W.$

Notes

$1 - \lim_{x\to 0} \frac{\sin x}{x} = 1$

$2 - \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$

$3 - \lim_{x\to 0} \frac{\tan x}{x} = 1$

$4 - \lim_{x\to 0} \frac{\sin ax}{ax} = 1$

5- Right and Left Limit

Example:- Is $\lim_{x\to 0}\frac{|x|}{x}$ exist at $x=0$?

Solve:

$$1-\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}(1)=1$$

$$2-\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1$$

$$\lim_{x\to 0^+}\frac{|x|}{x}\neq\lim_{x\to 0^-}\frac{|x|}{x}$$

Then Limit is not exist at $x=0$

Example:-

$$f(x)=\begin{cases}2x+1 & x>1\\ 5 & x=1\\ 7x^2-4 & x<1\end{cases}$$

Solve:

$$1-\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(2x+1)=\lim_{x\to 1^+}(2\cdot 1+1)=3$$

$$2-\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}(7x^2-4)=\lim_{x\to 1^-}(7\cdot 1-4)=3$$

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^-}f(x)$$

Then Limit is exist at $x=1$, and equal 3.

6- Hopital Rule

Using the Limit Hopital Rule for Ralition function at $\frac{\infty}{\infty}$ or $\frac{o}{o}$ such that derivative

Example:- Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$

Solve:

$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{o}{o}$ **then**

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{2x}{1} = 2\cdot 2 = 4$$

Example:- Find $\lim_{x\to 0} \frac{\sin x - x}{x^2}$

Solve:

$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \frac{o}{o}$ **then**

$\lim_{x\to 2} \frac{\sin x - x}{x^2} = \lim_{x\to 2} \frac{\cos x - 1}{2x} = \frac{0}{0}$ **then**

$$\lim_{x\to 2} \frac{\cos x - 1}{2x} = \lim_{x\to 2} \frac{-\sin x}{2} = \frac{0}{2} = 0$$

Example:- Find the following Limits by using Limit Hopital Rule

$$1 - \lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$$

Solve:

$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \frac{o}{o}$, then

$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \lim_{x\to 0} \frac{\left(\frac{1}{2}\right)(1+x)^{\frac{-1}{2}}-\frac{1}{2}}{2x} = \frac{0}{0}$, then

$$\lim_{x\to 0} \frac{\left(\frac{1}{2}\right)(1+x)^{\frac{-1}{2}}-\frac{1}{2}}{2x} = \lim_{x\to 0} \frac{-\left(\frac{1}{4}\right)(1+x)^{\frac{-3}{2}}}{2} = \frac{-1}{8}$$

$2 - \lim_{x\to 0} \frac{x-\sin x}{x^3}$

Solve:

$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \frac{o}{o}$, then

$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \frac{0}{0}$, then

$\lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{0}{0}$,then

$$\lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

$3 - \lim_{x\to \frac{\pi}{2}} [\sec x \cdot \tan x]$ *H.W.*

$4 - \lim_{x\to 0} \frac{\ln(x+1)-2x}{x^2}$ *H.W.*